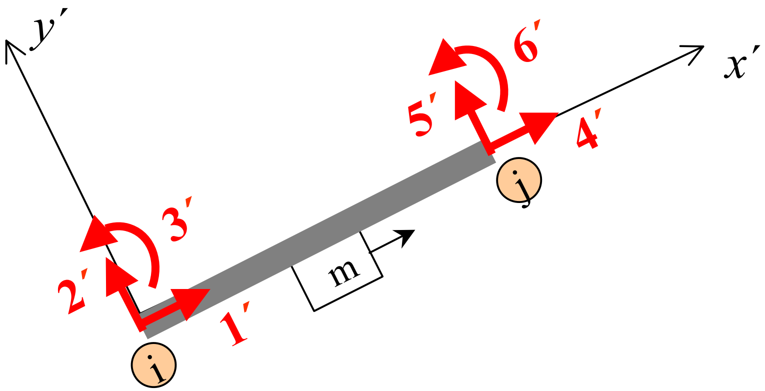
FRAME ANALYSIS USING THE STIFFNESS METHOD

- **Simple Frames**
  - **Frame-Member Stiffness Matrix**
  - **Displacement and Force Transformation Matrices**
  - **Frame-Member Global Stiffness Matrix**

- **Special Frames**
  - **Frame-Member Global Stiffness Matrix**

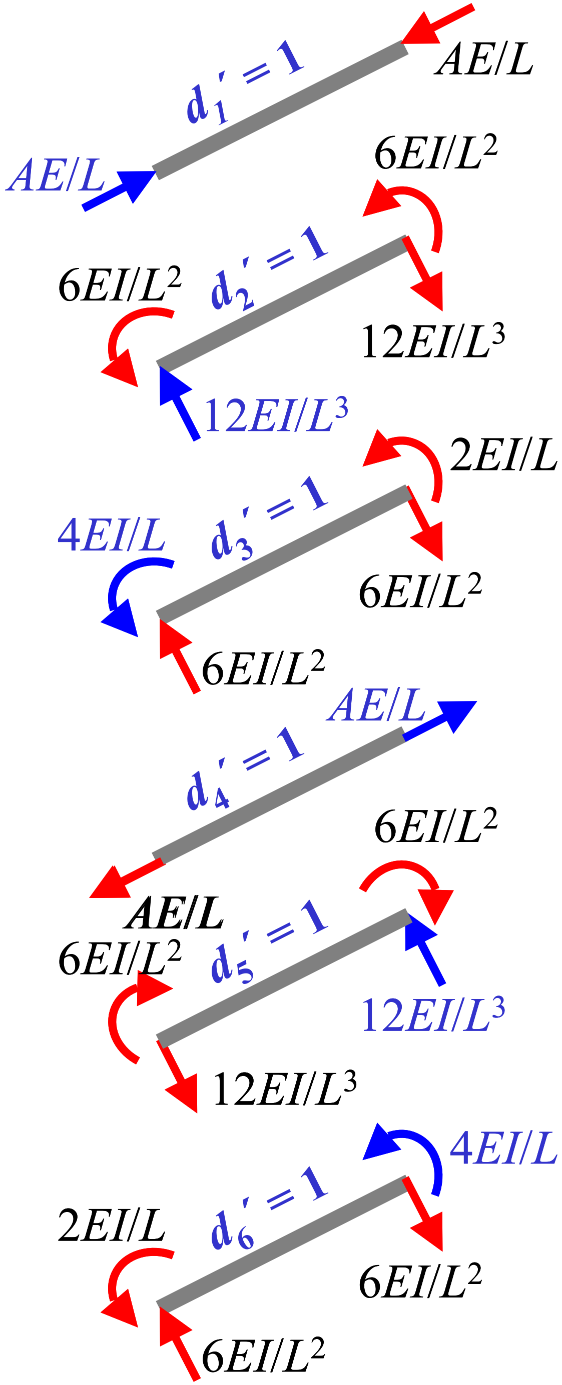
Simple Frames

Frame-Member Stiffness Matrix

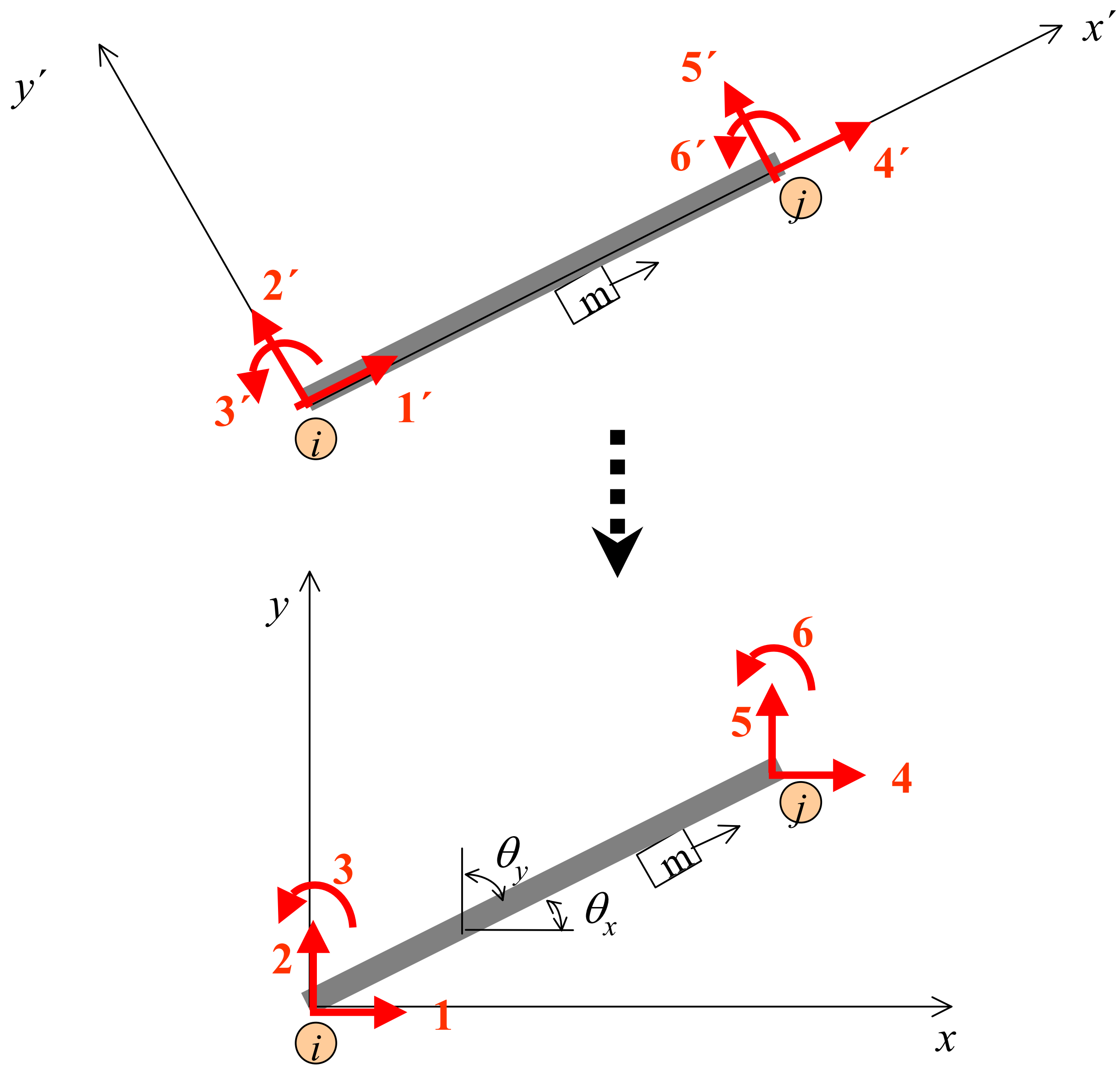


[k′]

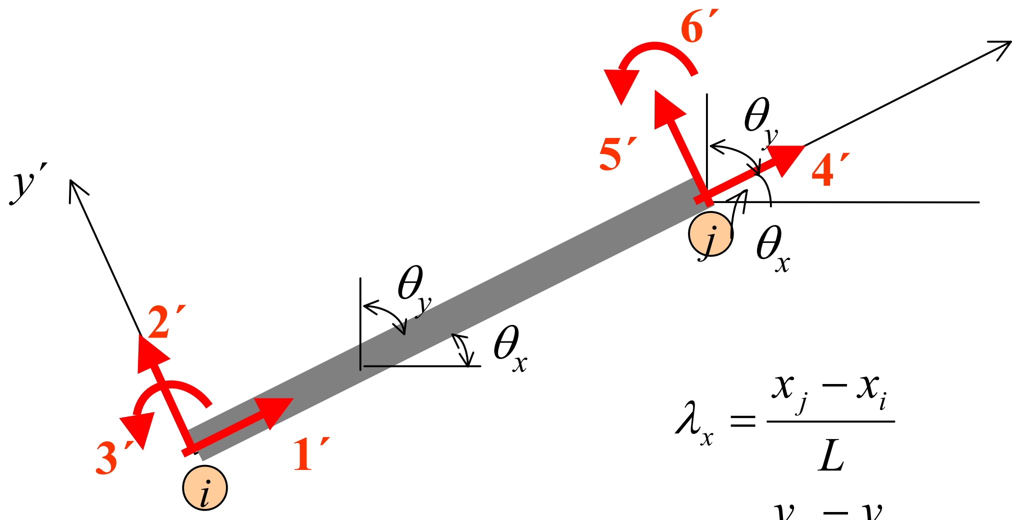
	1′	2′	3′	4′	5′	6′
1′	AE/L	0	0	$-AE/L$	0	0
2′	0	$12EI/L^3$	$6EI/L^2$	0	$-12EI/L^3$	$6EI/L^2$
3′	0	$6EI/L^2$	$4EI/L$	0	$-6EI/L^2$	$2EI/L$
4′	$-AE/L$	0	0	AE/L	0	0
5′	0	$-12EI/L^3$	$-6EI/L^2$	0	$12EI/L^3$	$-6EI/L^2$
6′	0	$6EI/L^2$	$2EI/L$	0	$-6EI/L^2$	$4EI/L$



Displacement and Force Transformation Matrices



Force Transformation



$$q_4 = q_{4'} \underbrace{\cos\theta_x}_{\lambda_x} - q_{5'} \underbrace{\cos\theta_y}_{\lambda_y}$$

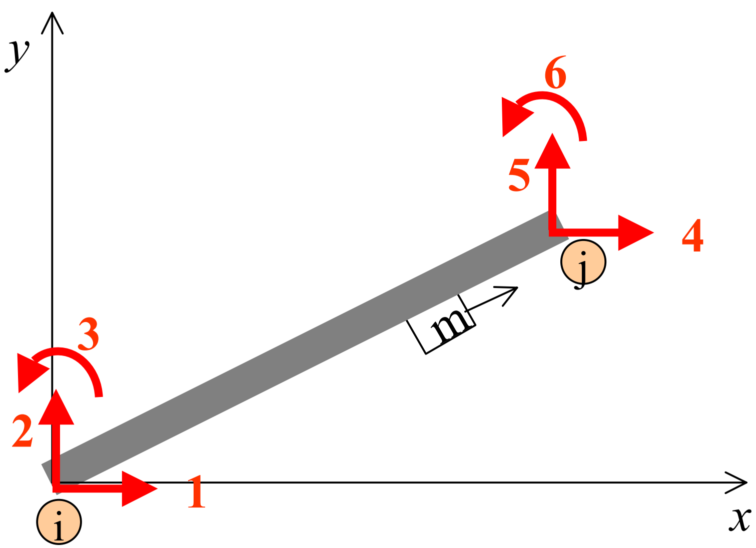
$$q_5 = q_{4'} \cos\theta_y + q_{5'} \cos\theta_x$$

$$q_6 = q_{6'}$$

$$\lambda_x = \frac{x_j - x_i}{L}$$

$$\lambda_y = \frac{y_j - y_i}{L}$$

$$\begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 \\ \lambda_y & \lambda_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 & 0 & 0 & 0 \\ \lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & -\lambda_y & 0 \\ 0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$$[q] = [T]^T [q']$$

$$[q] = [T]^T[q']$$

$$= [T]^T ([k'][d'] + [q'^F])$$

$$= [T]^T [k'][d'] + [T]^T [q'^F]$$

$$[q] = [T]^T [k'][T][d] + [T]^T [q'^F] = [k][d] + [q^F]$$

Therefore, $[k] = [T]^T [k'][T]$

$$[q^F] = [T]^T [q'^F]$$

$$[q] = [T]^T[q']$$

$$[d'] = [T][d]$$

$$[k] = [T]^T [k'][T]$$

Frame Member Global Stiffness Matrix

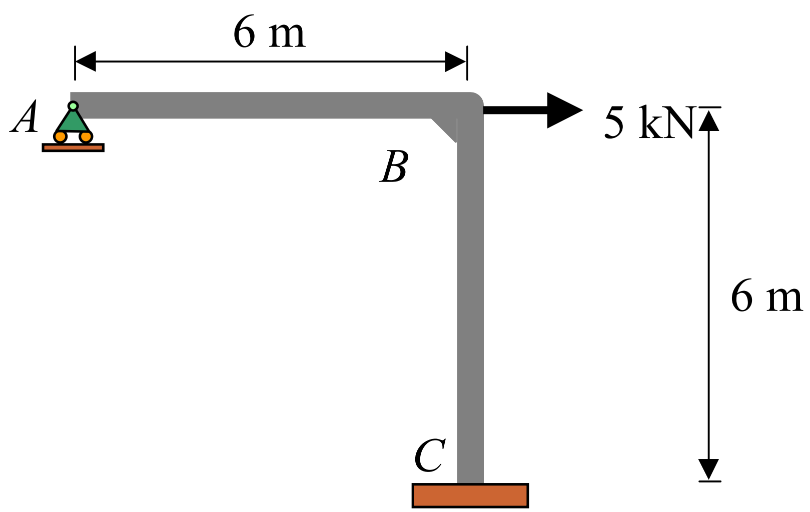
$$[q] = [T]^T[q'] = [T]^T\left([k'][d'] + [q'^F]\right) = [T]^T[k'][d'] + [T]^T[q'^F] = \underbrace{[T]^T[k'][T]}_{[k]}[d] + \underbrace{[T]^T[q'^F]}_{[q^F]}$$

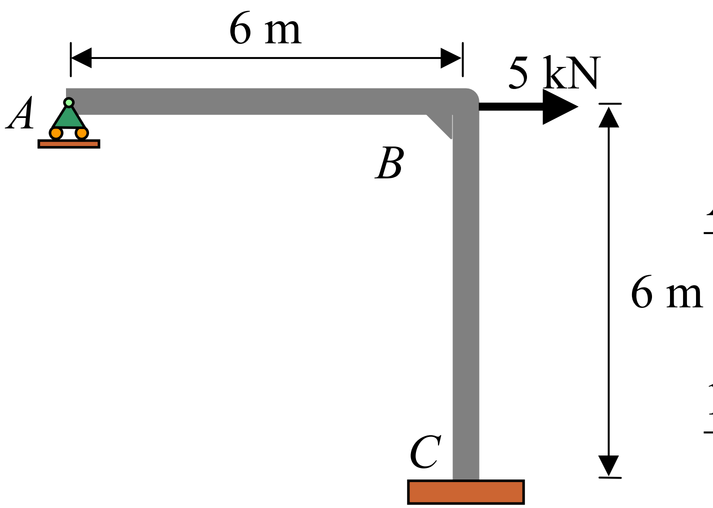
$$[k] = [T]^T[k'][T] =$$

$$\begin{array}{c} \\ U_i \\ \\ V_i \\ \\ M_i \\ \\ U_j \\ \\ V_j \\ \\ M_j \end{array}
\begin{array}{cccccc}
U_i & V_i & M_i & U_j & V_j & M_j \\
\left(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & -\frac{6EI}{L^2}\lambda_{iy} & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{iy} \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & \left(\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2\right) & \frac{6EI}{L^2}\lambda_{ix} & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & \frac{6EI}{L^2}\lambda_{ix} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{2EI}{L} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & \frac{6EI}{L^2}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \frac{6EI}{L^2}\lambda_{jy} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{jx} & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2\right) & -\frac{6EI}{L^2}\lambda_{jx} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{4EI}{L}
\end{array}$$

Example 1

For the frame shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**
$E = 200$ GPa, $I = 60(10^6)$ mm^4, $A = 600$ mm^2

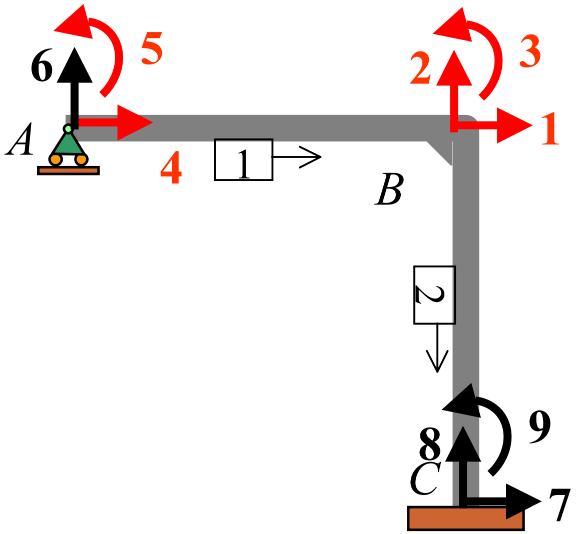




$$\frac{AE}{L} = \frac{(600\times10^{-6}\,\text{m}^2)(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})}{6\text{m}} = 20000\ \text{kN/m}$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{(6\text{m})^3} = 666.667\ \text{kN/m}$$

Global :



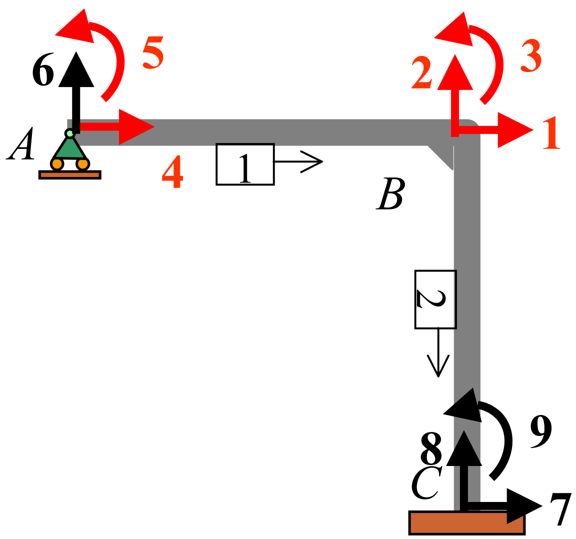
$$\frac{6EI}{L^2} = \frac{6(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{(6\text{m})^2} = 2000\ \text{kN}$$

$$\frac{4EI}{L} = \frac{4(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{6\text{m}} = 8000\ \text{kN}\bullet\text{m}$$

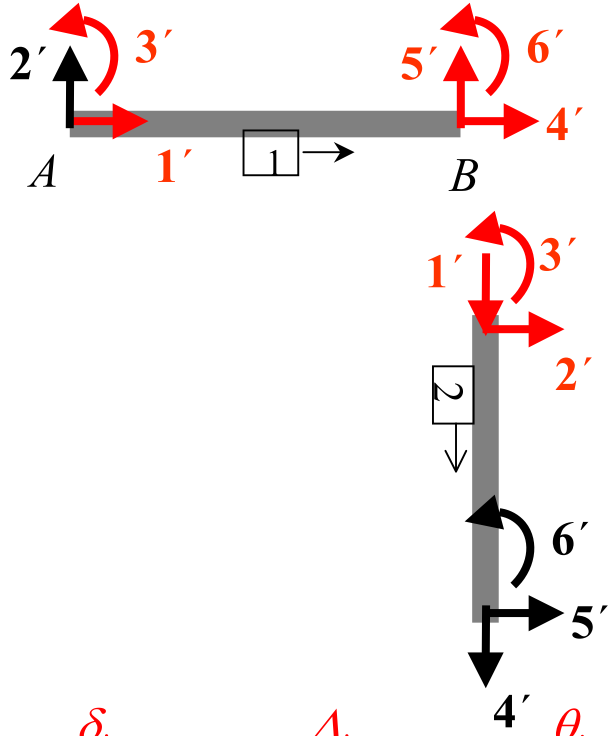
$$\frac{2EI}{L} = \frac{2(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{6\text{m}} = 4000\ \text{kN}\bullet\text{m}$$

Using Transformation Matrix:

Global :



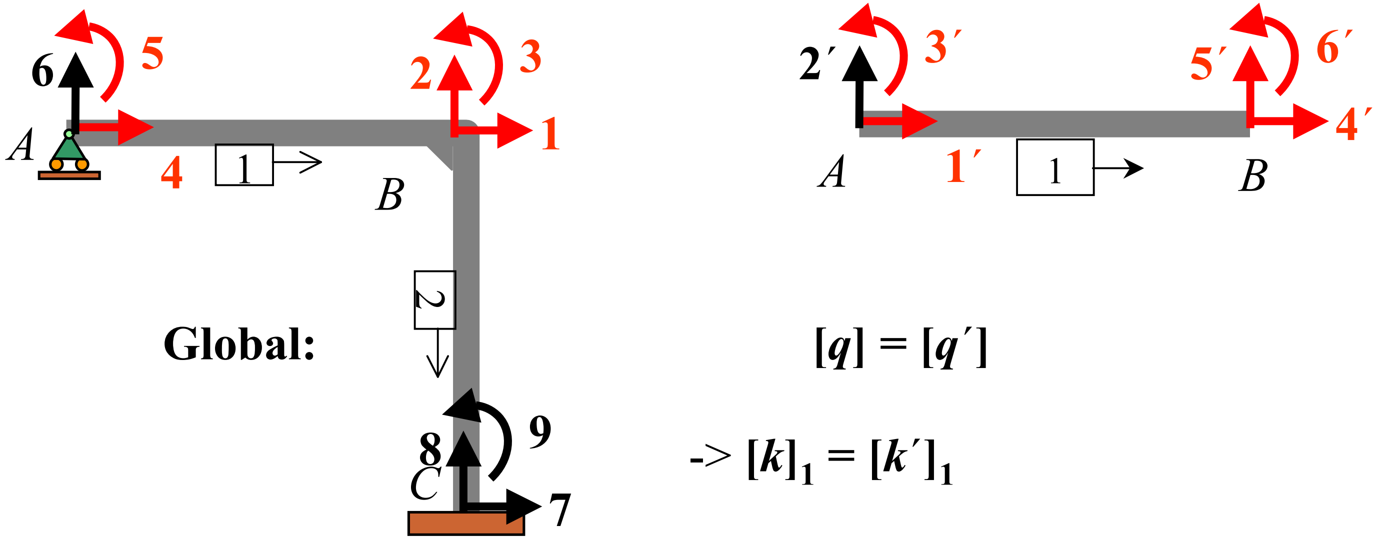
Local :



- **Member Stiffness Matrix**

$$
\begin{array}{c c}
 & \begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\
[\mathrm{k'}] = \begin{array}{c} N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array} &
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

Stiffness Matrix: Member 1



[q] = [q']

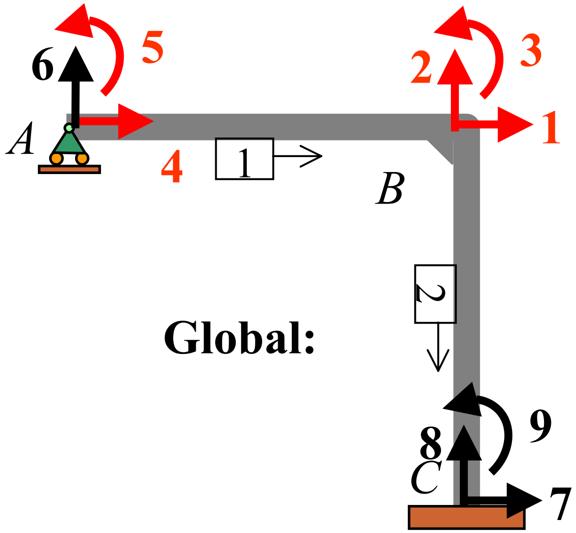
-> $[k]_1 = [k']_1$

$$[k]_1 = \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix}$$

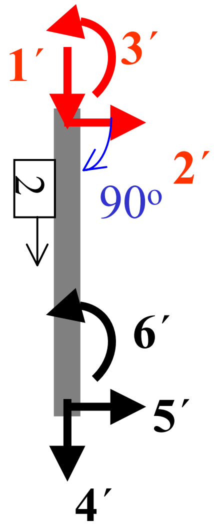
	4	6	5	1	2	3
4	20000	0	0	-20000	0	0
6	0	666.667	2000	0	-666.667	2000
5	0	2000	8000	0	-2000	4000
1	-20000	0	0	20000	0	0
2	0	-666.667	-2000	0	666.667	-2000
3	0	2000	4000	0	-2000	8000

Stiffness Matrix: Member 2

Global:



Local:



$\lambda_{ix} = \cos(-90^{o}) = 0$

$\lambda_{iy} = \sin(-90^{o}) = -1$

$\lambda_{jx} = \cos(-90^{o}) = 0$

$\lambda_{jy} = \sin(-90^{o}) = -1$

$[q]_2 = [\,T\,]^{T}[\,q'\,]_2$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{Bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{Bmatrix}$$

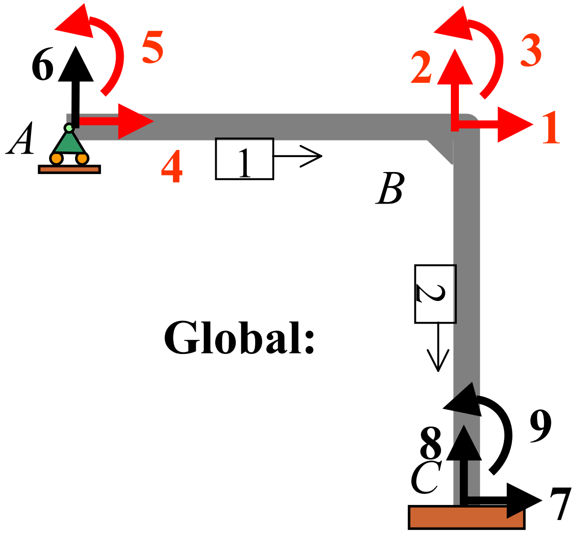
$[T]^{T}$

$$
[k']_2 = \begin{array}{c|cccccc|}
 & 1' & 2' & 3' & 4' & 5' & 6' \\
1' & 20000 & 0 & 0 & -20000 & 0 & 0 \\
2' & 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\
3' & 0 & 2000 & 8000 & 0 & -2000 & 4000 \\
4' & -20000 & 0 & 0 & 20000 & 0 & 0 \\
5' & 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\
6' & 0 & 2000 & 4000 & 0 & -2000 & 8000
\end{array}
$$

$[k]_2 = [\,T\,]^T[\,k'\,]_2[\,T\,]$

$$
[k]_2 = \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 7 & 8 & 9 \\
1 & 666.667 & 0 & 2000 & -666.667 & 0 & 2000 \\
2 & 0 & 20000 & 0 & 0 & -20000 & 0 \\
3 & 2000 & 0 & 8000 & -2000 & 0 & 4000 \\
7 & -666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\
8 & 0 & -20000 & 0 & 0 & 20000 & 0 \\
9 & 2000 & 0 & 4000 & -2000 & 0 & 8000
\end{array}
$$

Global Stiffness Matrix:



$[k]_1$

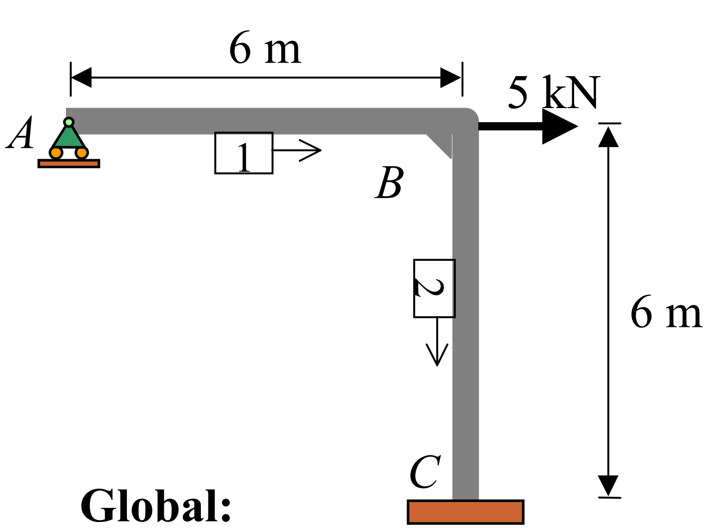
	4	6	5	1	2	3
4	20000	0	0	-20000	0	0
6	0	666.667	2000	0	-666.667	2000
5	0	2000	8000	0	-2000	4000
1	-20000	0	0	20000	0	0
2	0	-666.667	-2000	0	666.667	-2000
3	0	2000	4000	0	-2000	8000

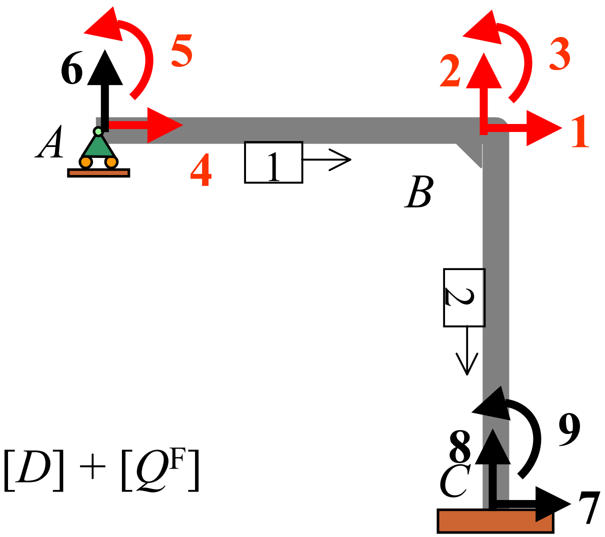
$[K]$

	4	5	1	2	3
4	20000	0	-20000	0	0
5	0	8000	0	-2000	4000
1	-20000	0	20666.667	0	2000
2	0	-2000	0	20666.667	-2000
3	0	4000	2000	-2000	16000

$[k]_2$

	1	2	3	7	8	9
1	666.667	0	2000	666.667	0	2000
2	0	20000	0	0	-20000	0
3	2000	0	8000	-2000	0	4000
7	-666.667	0	-2000	666.667	0	-2000
8	0	-20000	0	0	20000	0
9	2000	0	4000	2000	0	8000



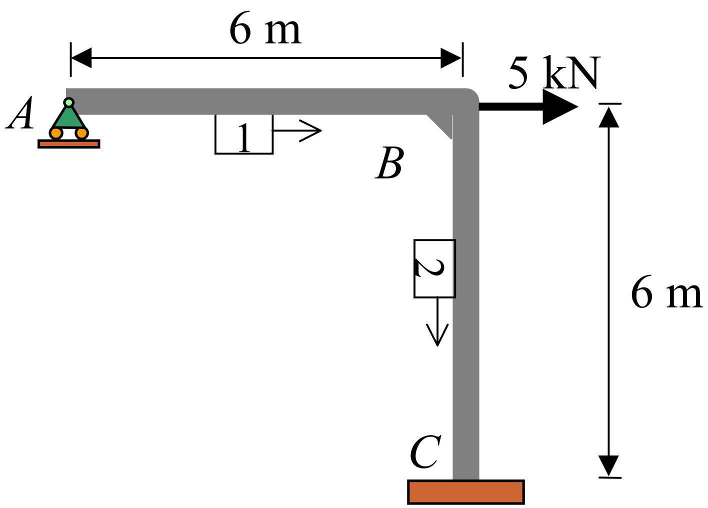


$$[Q] = [K][D] + [Q^F]$$

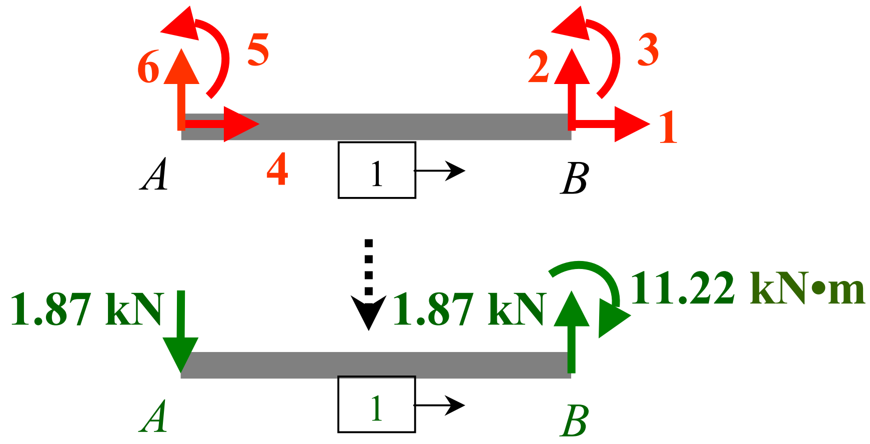
Global:

$$\begin{array}{c} \\ \begin{pmatrix} Q_4 = 0 \\ Q_5 = 0 \\ Q_1 = 5 \\ Q_2 = 0 \\ Q_3 = 0 \end{pmatrix} \end{array} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 20000 & 0 & -20000 & 0 & 0 \\ 0 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 20666.667 & 0 & 2000 \\ 0 & -2000 & 0 & 20666.667 & -2000 \\ 0 & 4000 & 2000 & -2000 & 16000 \end{pmatrix} \end{array} \begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.01316 \text{ m} \\ 9.199(10^{-4}) \text{ rad} \\ 0.01316 \text{ m} \\ -9.355(10^{-5}) \text{ m} \\ -1.887(10^{-3}) \text{ rad} \end{pmatrix}$$

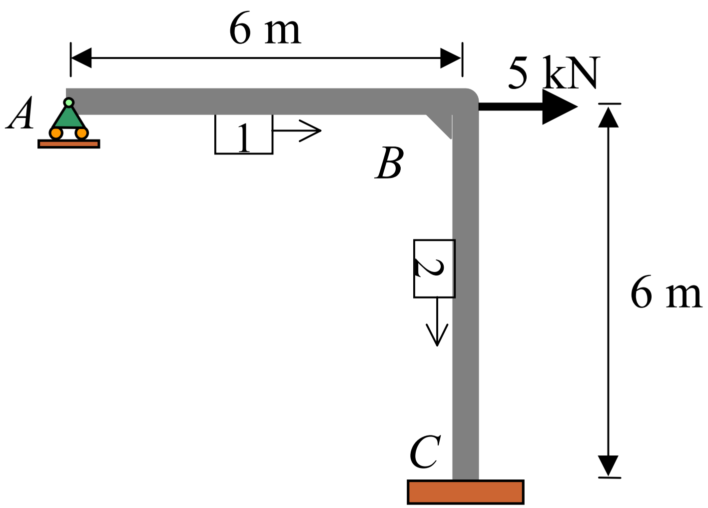


Member 1

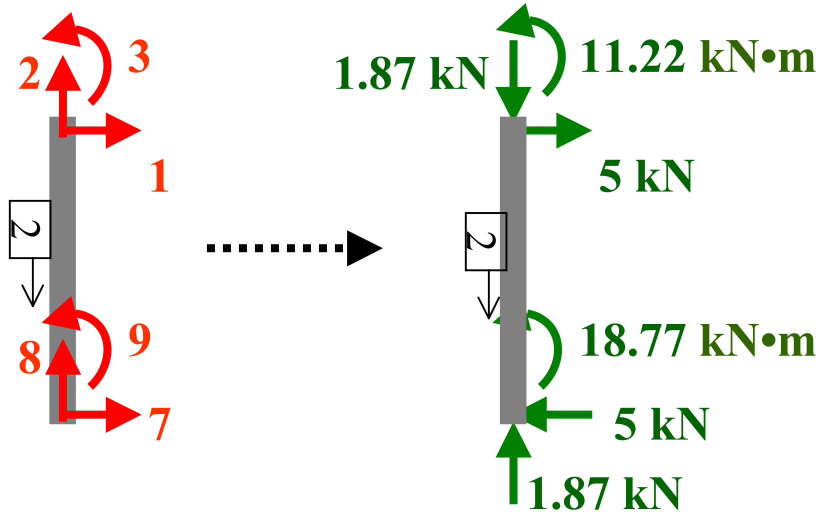


$$[q]_1 = [k]_1[d]_1 + [q^F]_1$$

$$\begin{pmatrix} q_4 \\ q_6 \\ q_5 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array} \begin{pmatrix} D_4 = 0.01316 \\ D_6 = 0 \\ D_5 = 9.199(10^{-4}) \\ D_1 = 0.01316 \\ D_2 = -9.355(10^{-5}) \\ D_3 = -1.887(10^{-3}) \end{pmatrix} = \begin{pmatrix} 0 \\ -1.87 \\ 0 \\ 0 \\ 1.87 \\ -11.22 \end{pmatrix}$$

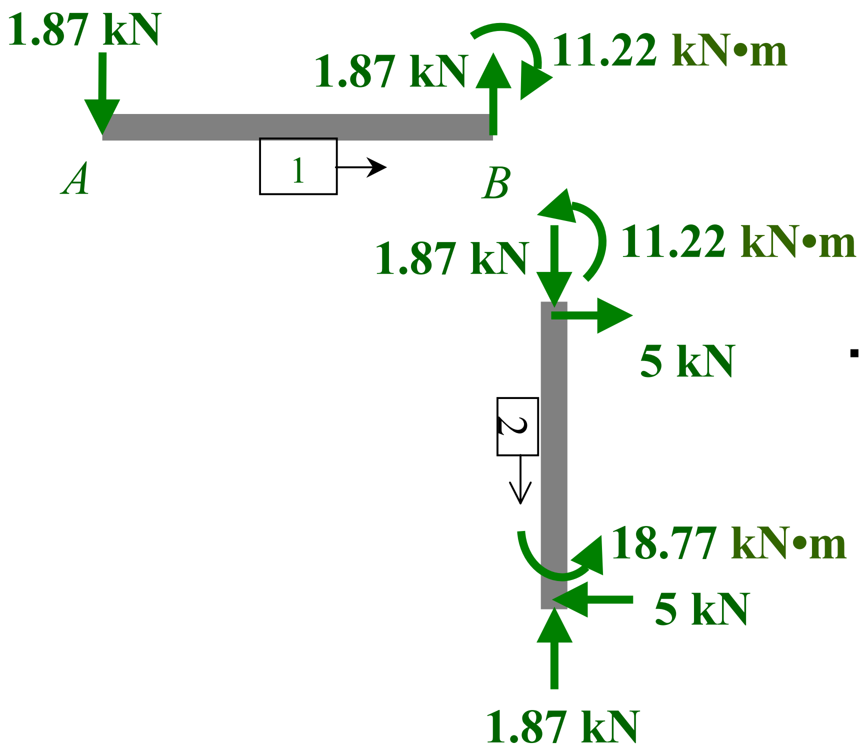


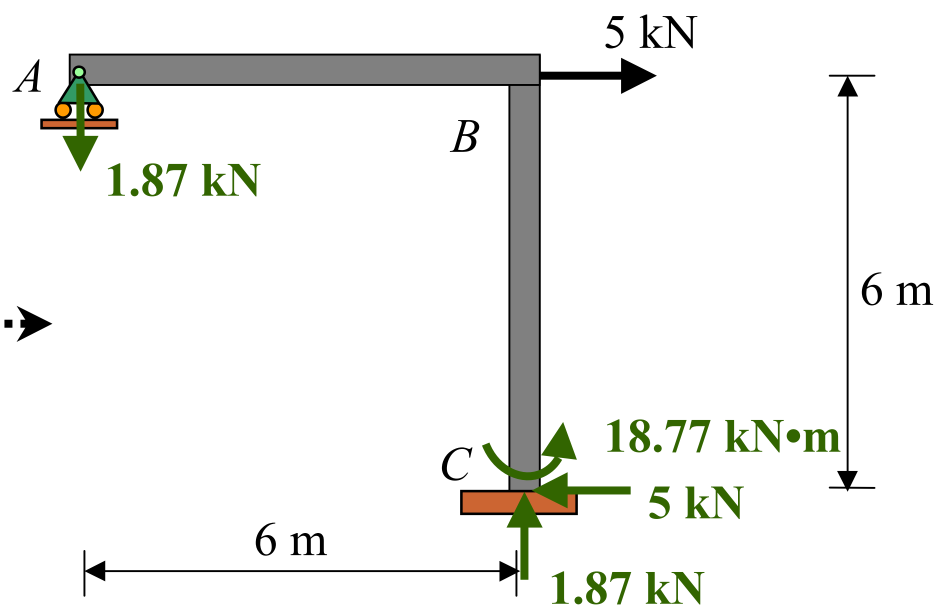
Member 2

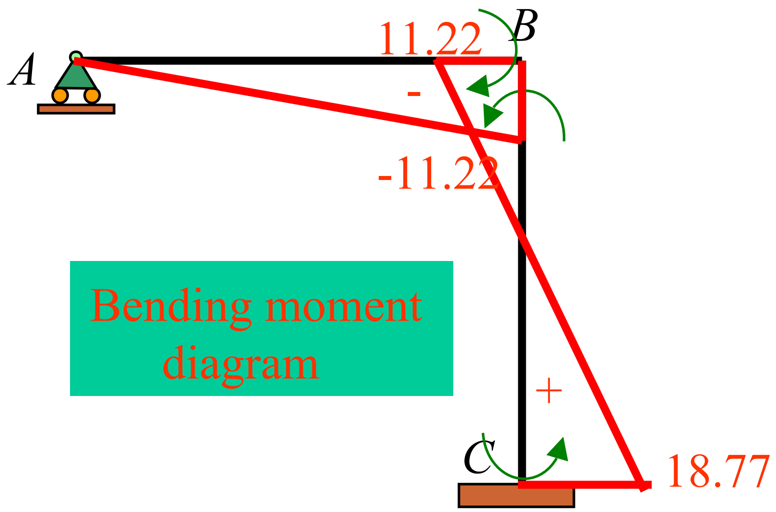


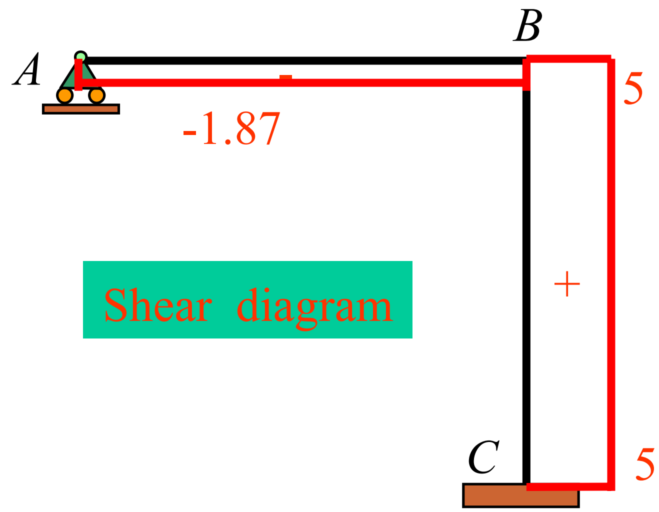
$$[q]_2 = [k]_2[d]_2 + [q^F]_2$$

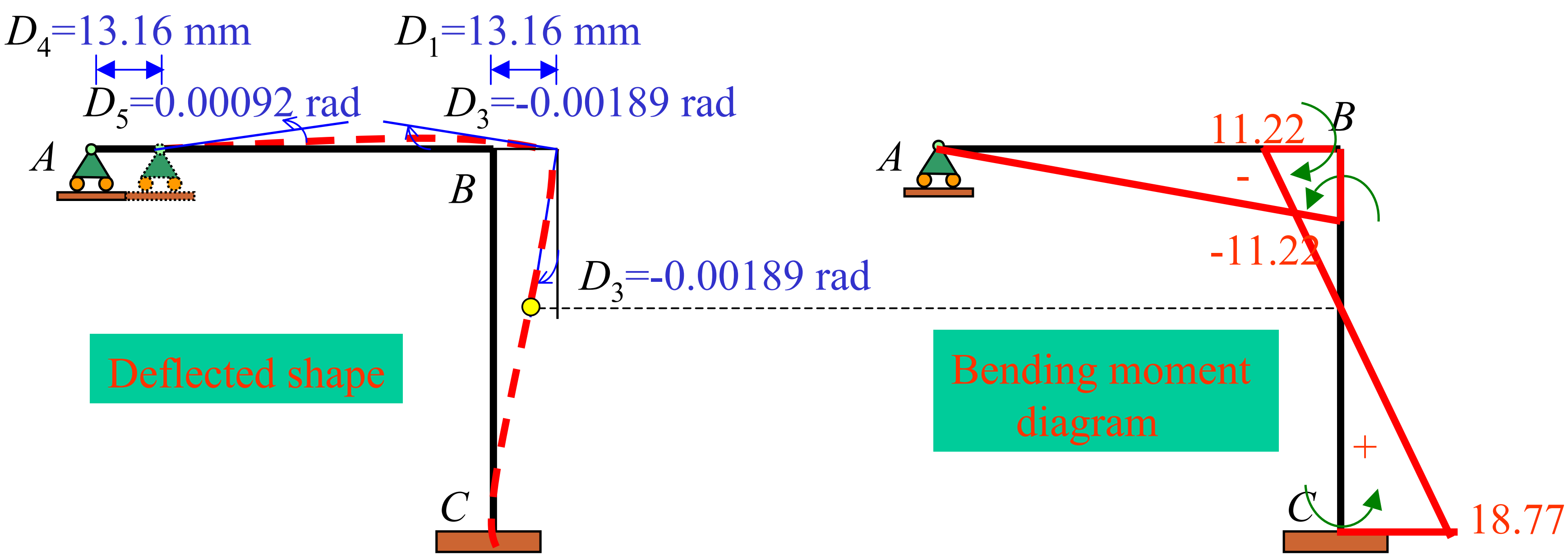
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{pmatrix} \overset{1}{666.667} & \overset{2}{0} & \overset{3}{2000} & \overset{7}{-666.667} & \overset{8}{0} & \overset{9}{2000} \\ 0 & 20000 & 0 & 0 & -20000 & 0 \\ 2000 & 0 & 8000 & -2000 & 0 & 4000 \\ -666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\ 0 & -20000 & 0 & 0 & 20000 & 0 \\ 2000 & 0 & 4000 & -2000 & 0 & 8000 \end{pmatrix} \begin{pmatrix} D_1 = 0.01316 \\ D_2 = -9.355(10^{-5}) \\ D_3 = -1.887(10^{-3}) \\ D_7 = 0 \\ D_8 = 0 \\ D_9 = 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1.87 \\ 11.22 \\ -5 \\ 1.87 \\ 18.77 \end{pmatrix}$$


1.87 kN
1.87 kN
11.22 kN•m
A
1
B
1.87 kN
11.22 kN•m
5 kN
2
18.77 kN•m
5 kN
1.87 kN

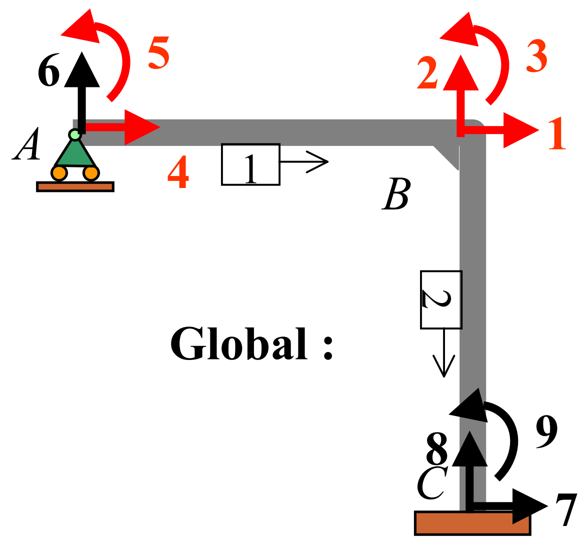

A
5 kN
B
1.87 kN
6 m
C
18.77 kN•m
5 kN
6 m
1.87 kN


A
11.22
B
-
-11.22
Bending moment diagram
+
C
18.77


A
B
-
5
-1.87
Shear diagram
+
C
5



$$\begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.01316 \text{ m} \\ 9.199(10^{-4}) \text{ rad} \\ 0.01316 \text{ m} \\ -9.355(10^{-5}) \text{ m} \\ -1.887(10^{-3}) \text{ rad} \end{pmatrix}$$



Example 2

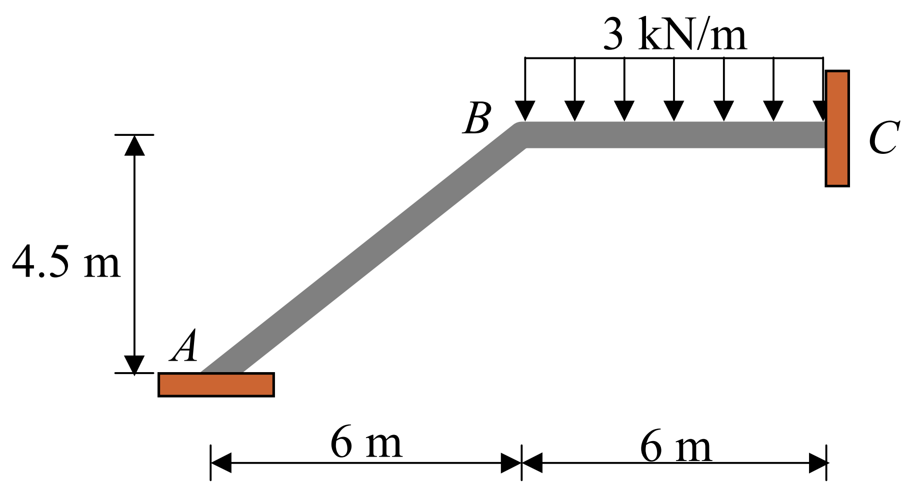
For the beam shown, use the stiffness method to:

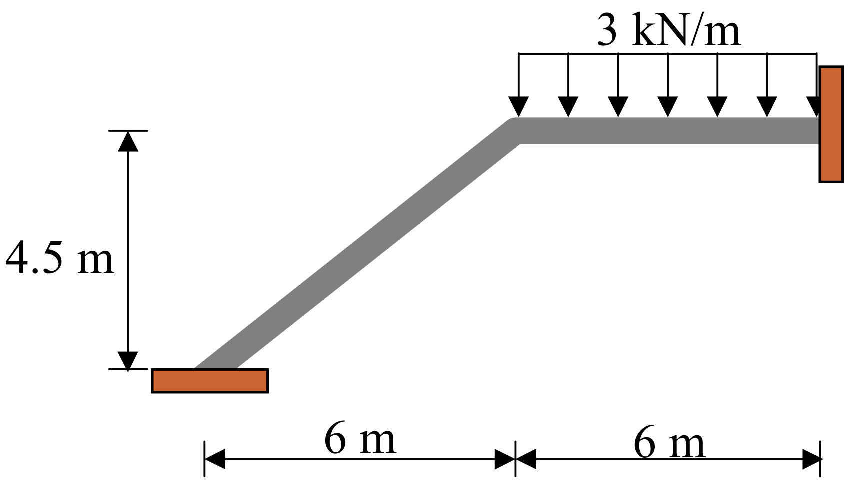
(a) Determine the **deflection** and **rotation** at ***B***

(b) Determine all the reactions at supports

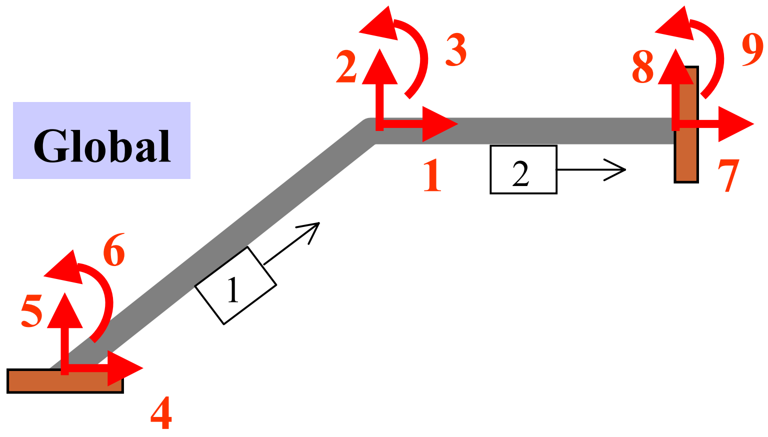
(c) Draw the **quantitative shear** and **bending moment diagrams.**

$E = 200$ GPa, $I = 60(10^6)$ mm^4, $A = 600$ mm^2 for each member.

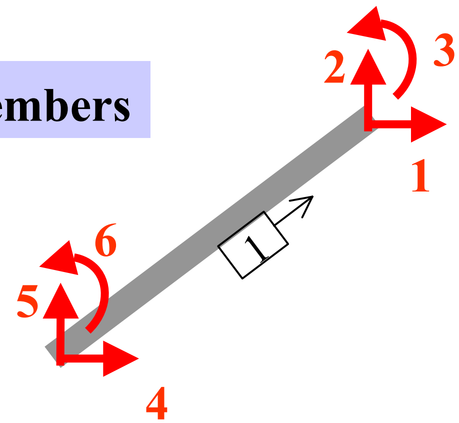


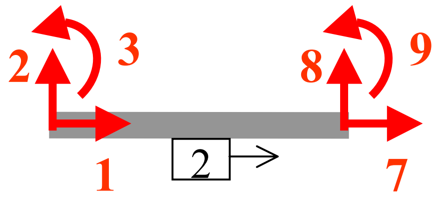


Global

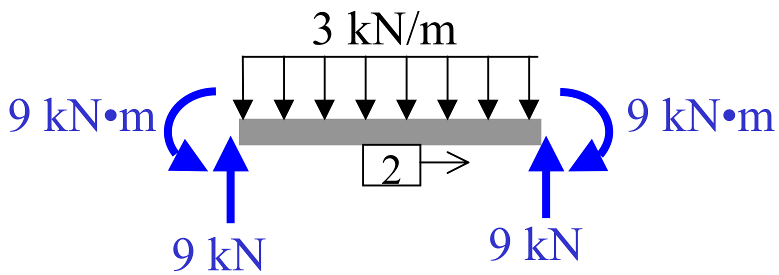


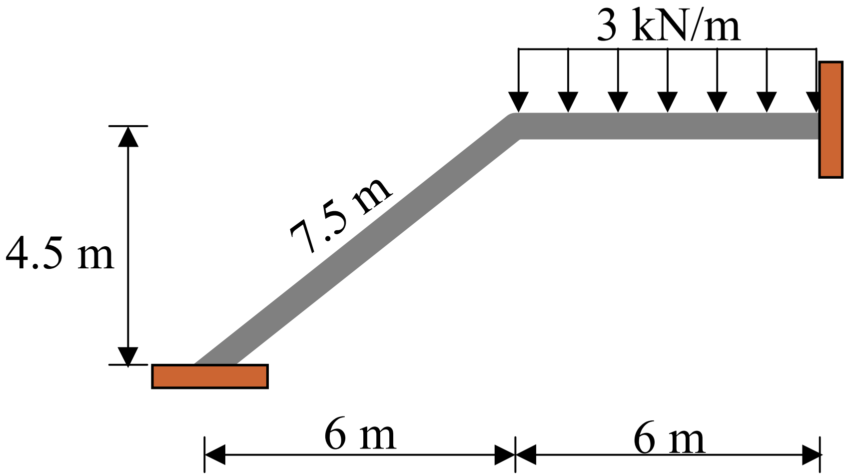
Members



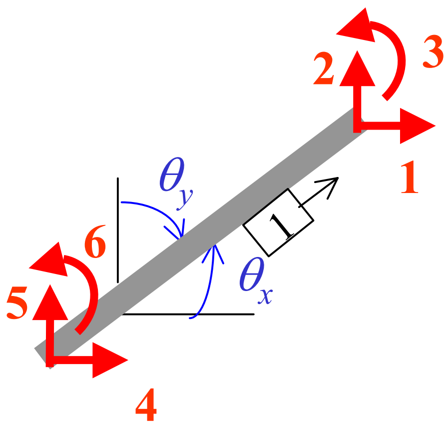


[FEM]





Member 1:



$\lambda_x = \cos\,\theta_x = 6/7.5 = \ 0.8$

$\lambda_y = \cos\,\theta_y = 4.5/7.5 = \ 0.6$

$$\frac{AE}{L} = \frac{(600\times10^{-6}\ m^2)(200\times10^6\ kN/m^2)}{7.5\ m}$$

$$= 16000\ kN/m$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(7.5\ m)^3}$$

$$= 341.33\ kN/m$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(7.5\ m)^2}$$

$$= 1280\ kN$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{7.5\ m}$$

$$= 6400\ kN\bullet m$$

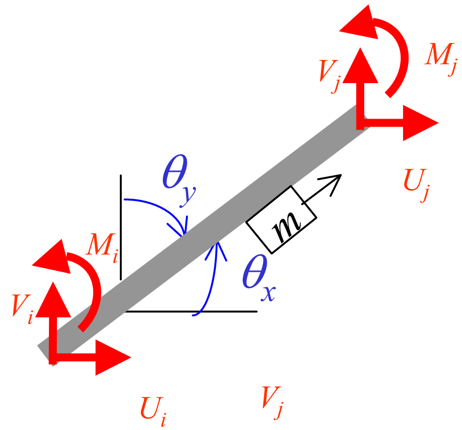
$$\frac{2EI}{L} = \frac{2(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{7.5\ m}$$

$$= 3200\ kN\bullet m$$

Member m:

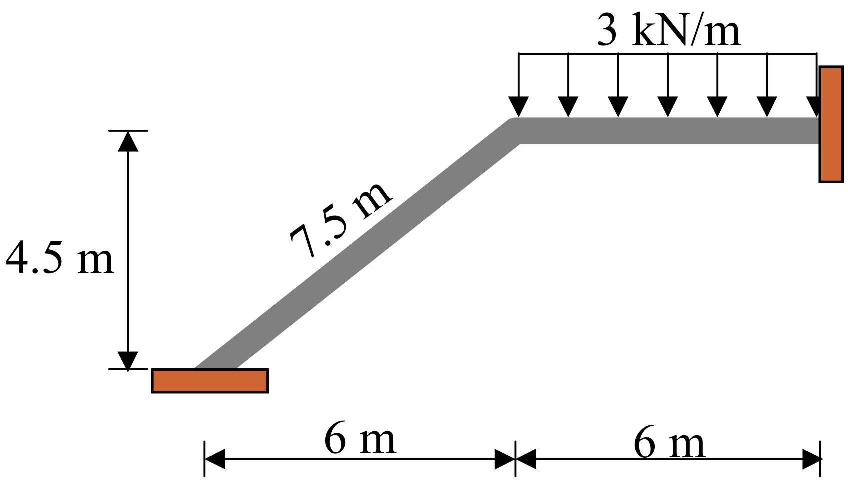
$$\lambda_x = \cos\,\theta_x$$

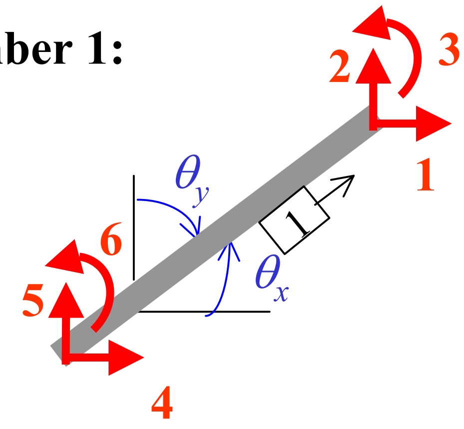
$$\lambda_y = \cos\,\theta_y$$



$$[\,\boldsymbol{k_m}\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{k'}\,][\boldsymbol{T}\,] =$$

$$\begin{array}{c} \\ U_i \\ V_i \\ M_i \\ U_j \\ V_j \\ M_j \end{array}
\begin{array}{cccccc}
U_i & V_i & M_i & U_j & V_j & M_j \\
\left(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & -\frac{6EI}{L^2}\lambda_{iy} & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{iy} \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & \left(\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2\right) & \frac{6EI}{L^2}\lambda_{ix} & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & \frac{6EI}{L^2}\lambda_{ix} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{2EI}{L} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & \frac{6EI}{L^2}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \frac{6EI}{L^2}\lambda_{jy} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{jx} & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2\right) & -\frac{6EI}{L^2}\lambda_{jx} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{4EI}{L}
\end{array}$$



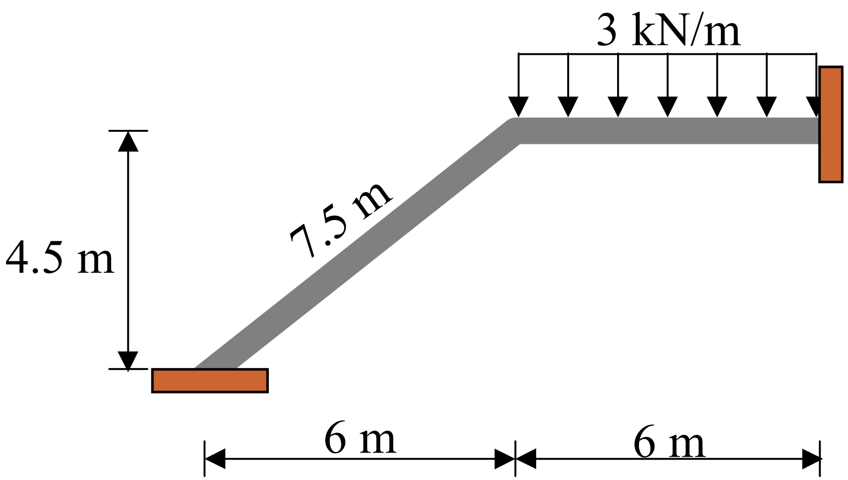


$\lambda_x = \cos\theta_x = 6/7.5 = 0.8$

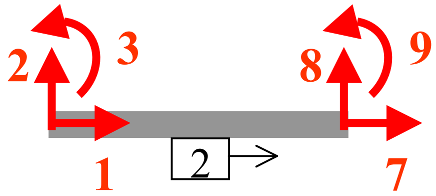
$\lambda_y = \cos\theta_y = 4.5/7.5 = 0.6$

$[k_1] =$

	4	5	6	1	2	3
4	10362.879	7516.162	-768	-10362.879	-7516.162	-768
5	7516.162	5978.451	1024	-7516.162	-5978.451	1024
6	-768	1024	6400	768	-1024	3200
1	-10362.879	-7516.162	768	10362.879	7516.162	768
2	-7516.162	-5978.451	-1024	7516.162	5978.451	-1024
3	-768	1024	3200	768	-1024	6400



Member 2:



$\lambda_x = \cos 0^{o} = 1.0,\ \lambda_y = \cos 90^{o} = 0$

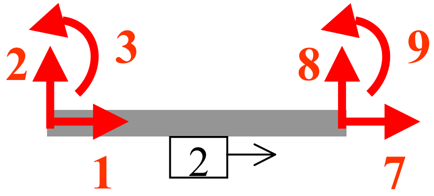
$$\frac{AE}{L} = \frac{(600\times10^{-6}\ m^2)(200\times10^6\ kN/m^2)}{6\ m}$$
$$= 20000\ kN/m$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(6\ m)^3}$$
$$= 666.667\ kN/m$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(6\ m)^2}$$
$$= 2000\ kN$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{6\ m}$$
$$= 8000\ kN \bullet m$$

$$\frac{2EI}{L} = \frac{2(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{6\ m}$$
$$= 4000\ kN \bullet m$$



$$[k_2] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{pmatrix} \end{array}$$

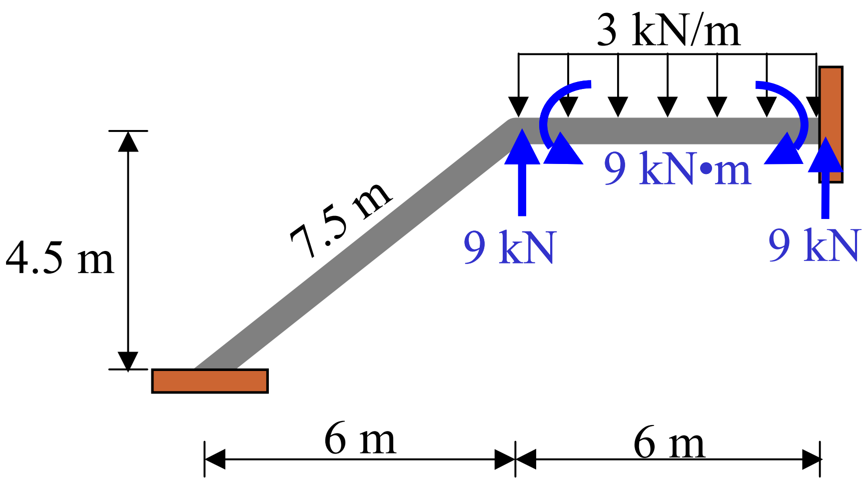
$$[k_2] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array}$$

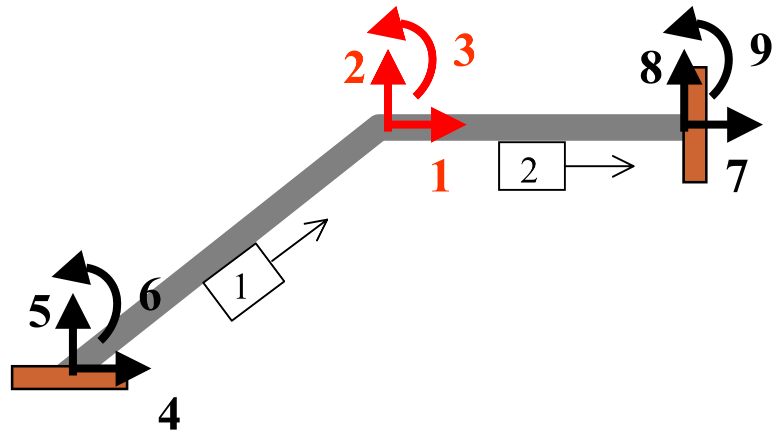
$[k_1] =$

	4	5	6	1	2	3
4	10362.879	7516.162	-768	-10362.879	-7516.162	-768
5	7516.162	5978.451	1024	-7516.162	-5978.451	1024
6	-768	1024	6400	768	-1024	3200
1	-10362.879	-7516.162	768	10362.879	7516.162	768
2	-7516.162	-5978.451	-1024	7516.162	5978.451	-1024
3	-768	1024	3200	768	-1024	6400

$[k_2] =$

	1	2	3	7	8	9
1	20000	0	0	- 20000	0	0
2	0	666.667	2000	0	- 666.667	2000
3	0	2000	8000	0	- 2000	4000
7	-20000	0	0	20000	0	0
8	0	-666.667	-2000	0	666.667	-2000
9	0	2000	4000	0	- 2000	8000



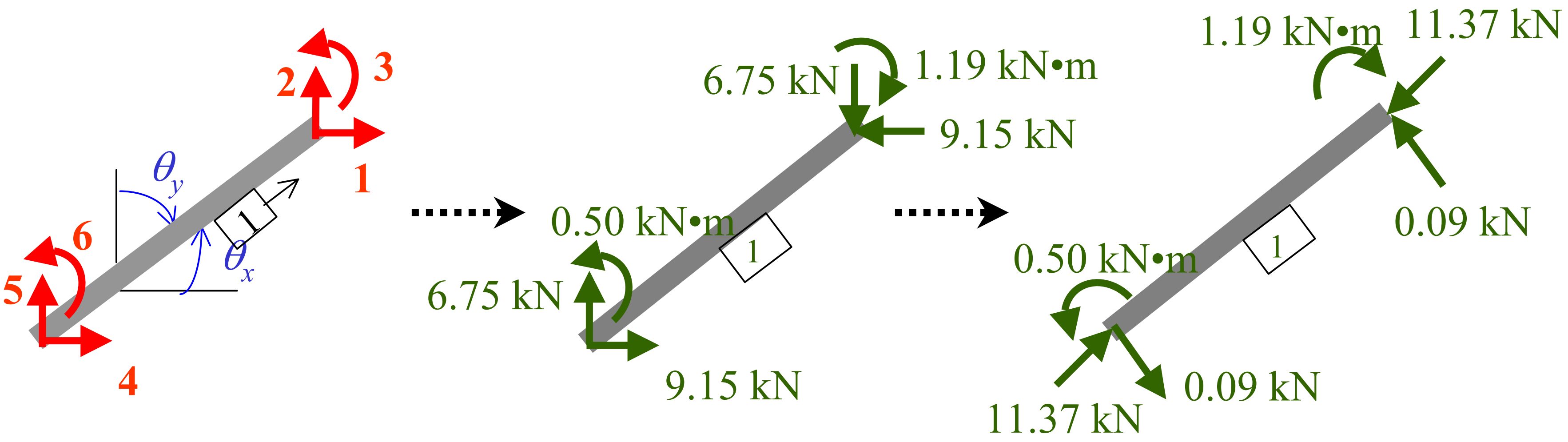


Global:

$$\begin{bmatrix} \cancelto{0}{Q_1} \\ \cancelto{0}{Q_2} \\ \cancelto{0}{Q_3} \end{bmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} 1 & 2 & 3 \\ \begin{bmatrix} 30362.9 & 7516.16 & 768 \\ 7516.16 & 6645.12 & 976 \\ 768 & 976 & 14400 \end{bmatrix} \end{matrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 4.575(10^{-4}) \text{ m} \\ -1.794(10^{-3}) \text{ m} \\ -5.278(10^{-4}) \text{ rad} \end{bmatrix}$$

Member 1:

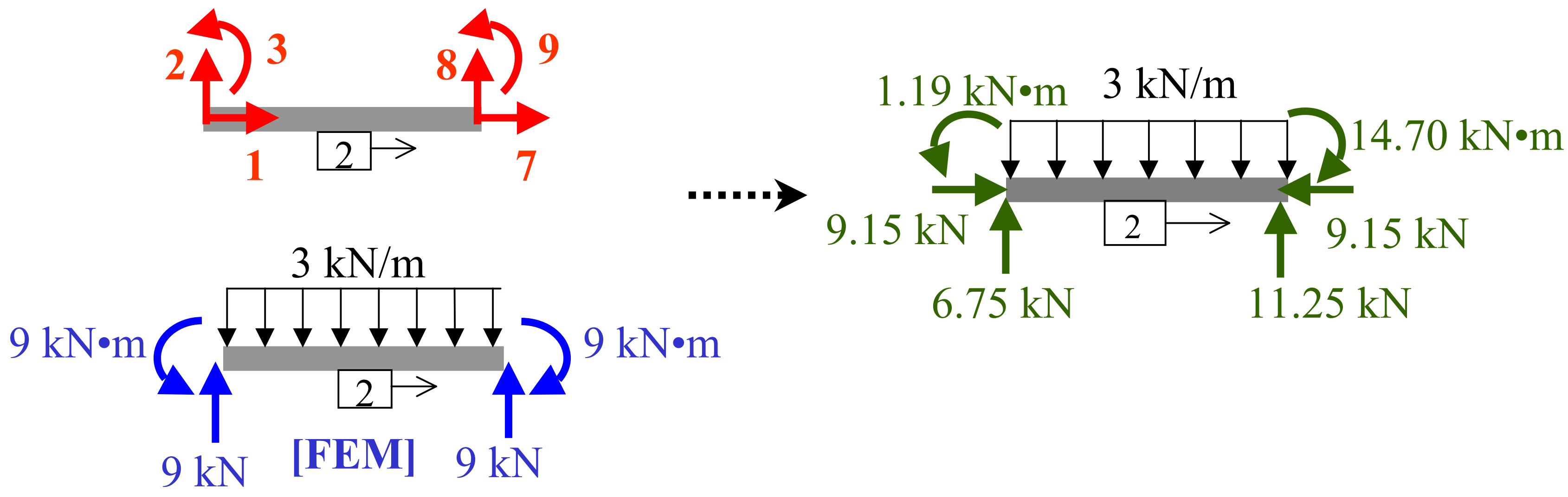


$$\lambda_x = \cos\,\theta_x = 6/7.5 = 0.8$$

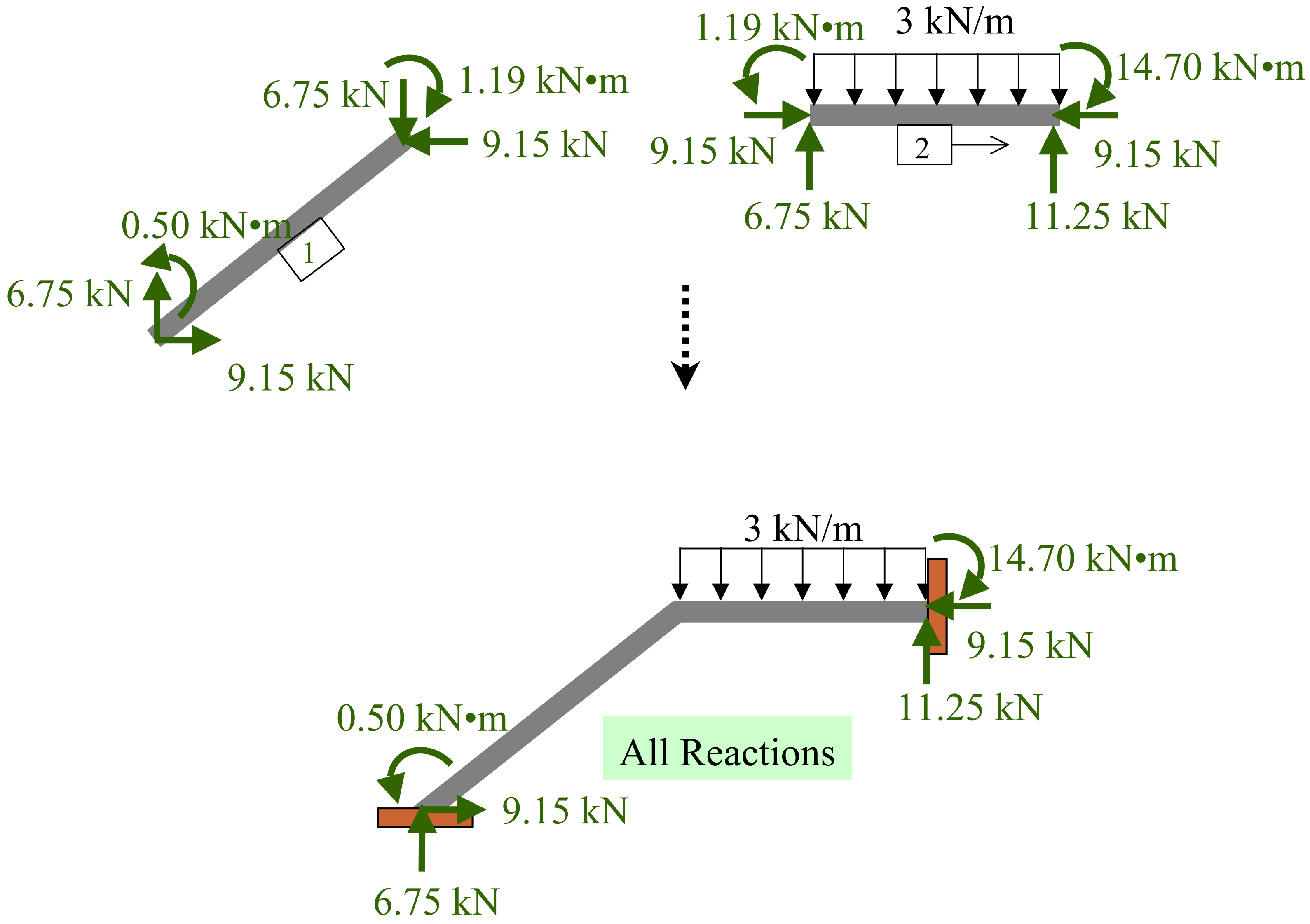
$$\lambda_y = \cos\,\theta_y = 4.5/7.5 = 0.6$$

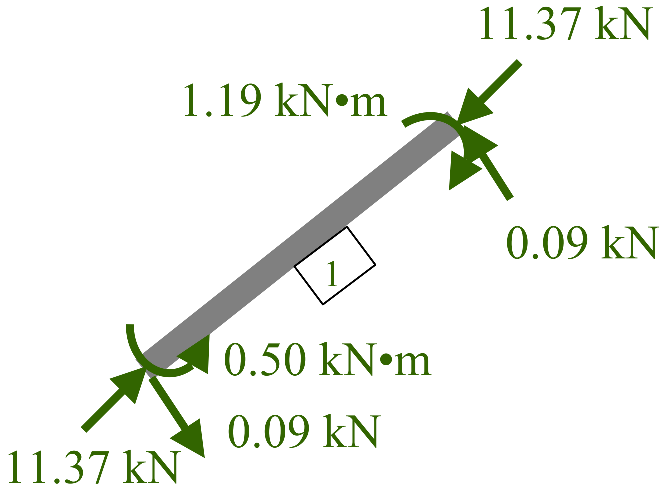
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{matrix} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix} \overset{\begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix}}{\begin{pmatrix} & & & & & \\ & & & & & \\ & & \boldsymbol{k_1} & & & \\ & & & & & \\ & & & & & \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_1 = 4.575(10^{-4}) \\ D_2 = -1.794(10^{-3}) \\ D_3 = -5.278(10^{-4}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9.15 \\ 6.75 \\ 0.50 \\ -9.15 \\ -6.75 \\ -1.19 \end{pmatrix}$$

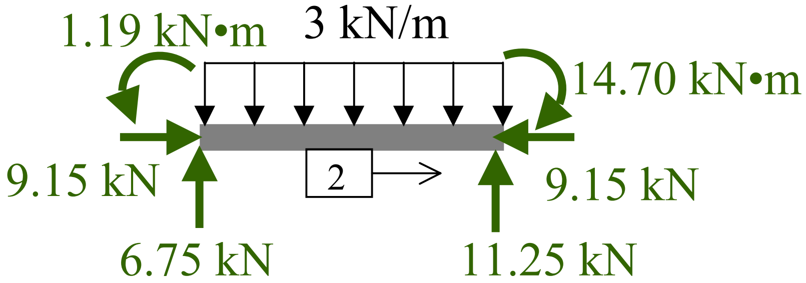
Member 2:



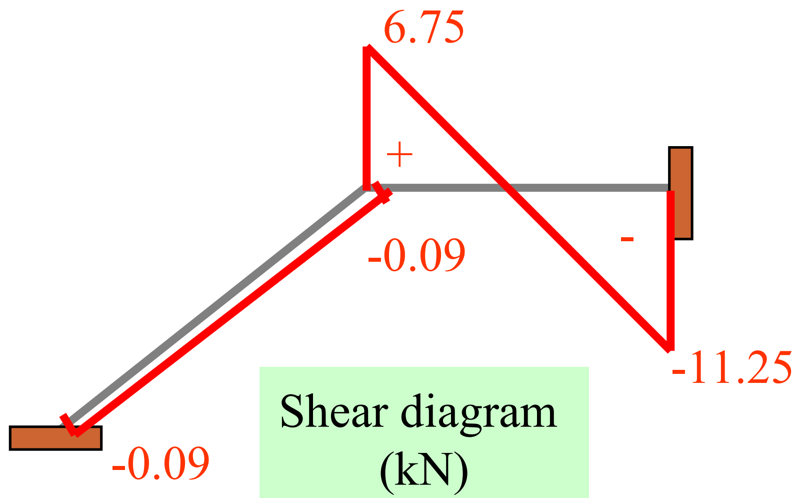
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{matrix} \begin{matrix} 1 \quad 2 \quad 3 \quad 7 \quad 8 \quad 9 \\ \begin{pmatrix} & & & & & \\ & & & & & \\ & & \boldsymbol{k_2} & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{matrix} \begin{pmatrix} D_1 = 4.575(10^{-4}) \\ D_2 = -1.794(10^{-3}) \\ D_3 = -5.278(10^{-4}) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \\ 9 \\ 0 \\ 9 \\ -9 \end{pmatrix} = \begin{pmatrix} 9.15 \\ 6.75 \\ 1.19 \\ -9.15 \\ 11.25 \\ -14.70 \end{pmatrix}$$


6.75 kN
1.19 kN•m
9.15 kN
0.50 kN•m
1
6.75 kN
9.15 kN
1.19 kN•m
3 kN/m
14.70 kN•m
9.15 kN
2
9.15 kN
6.75 kN
11.25 kN
3 kN/m
14.70 kN•m
9.15 kN
11.25 kN
0.50 kN•m
All Reactions
9.15 kN
6.75 kN

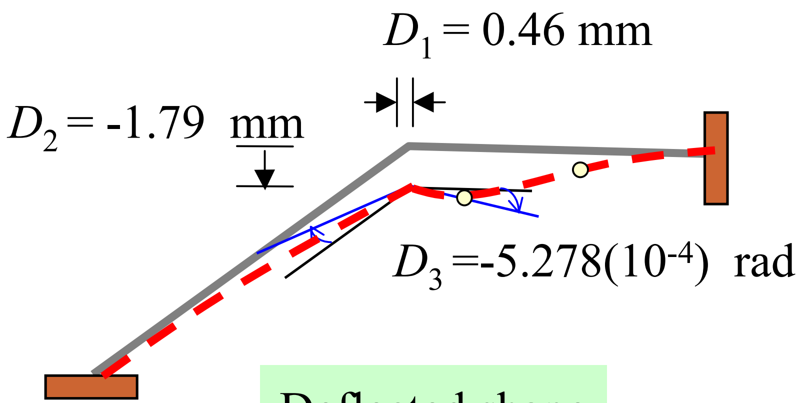




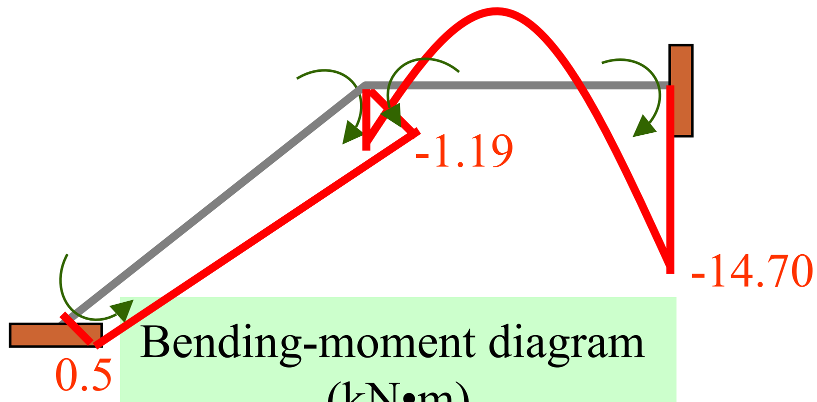
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 4.575(10^{-4})\ \text{m} \\ -1.794(10^{-3})\ \text{m} \\ -5.278(10^{-4})\ \text{rad} \end{pmatrix}$$



Shear diagram (kN)



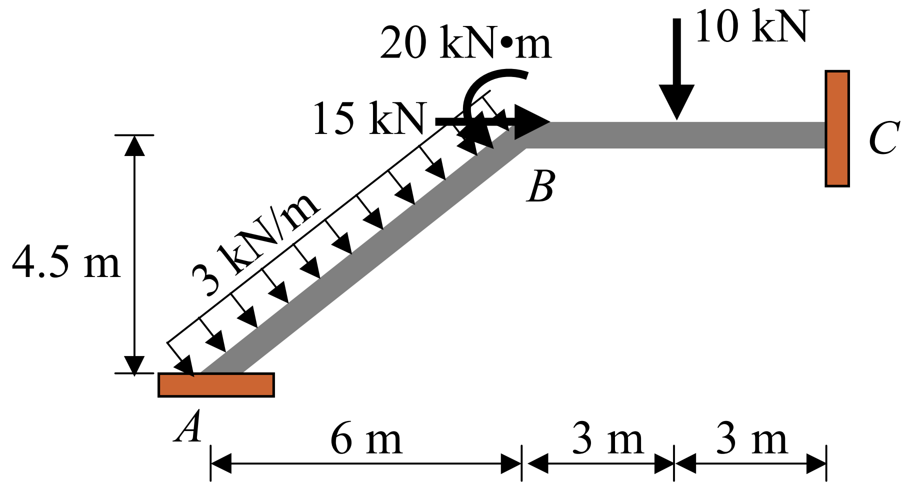
Deflected shape

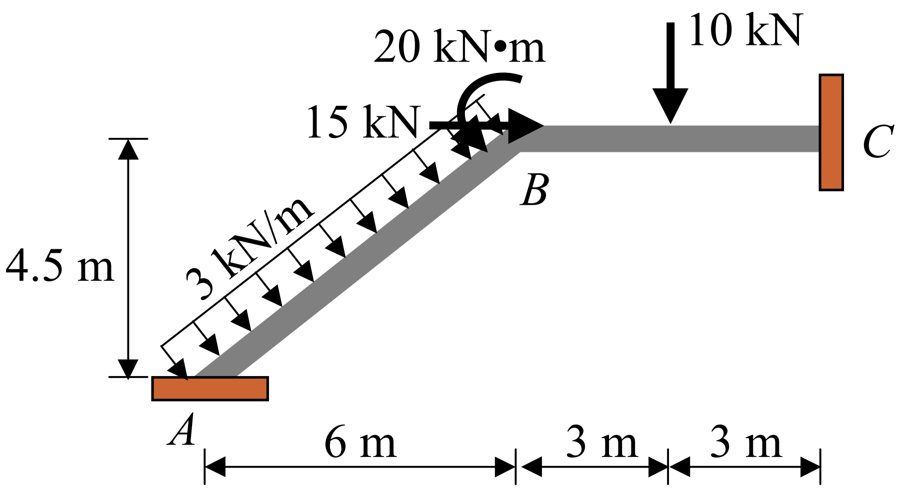


Bending-moment diagram (kN•m)

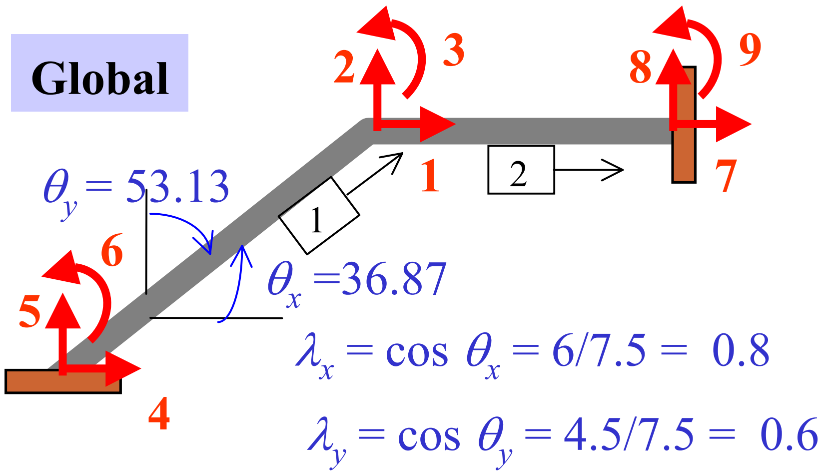
Example 3

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**
$E = 200$ GPa, $I = 60(10^6)$ mm^4, $A = 600$ mm^2 for each member.

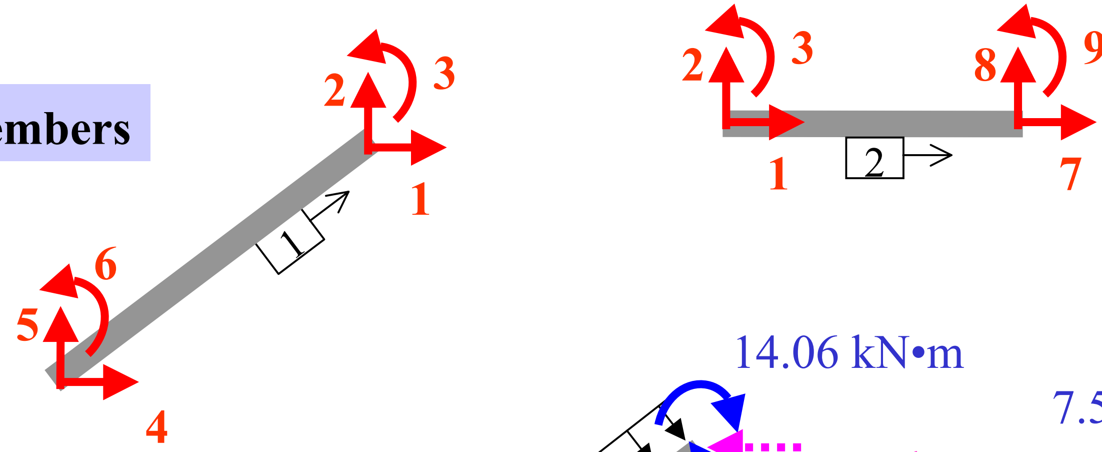




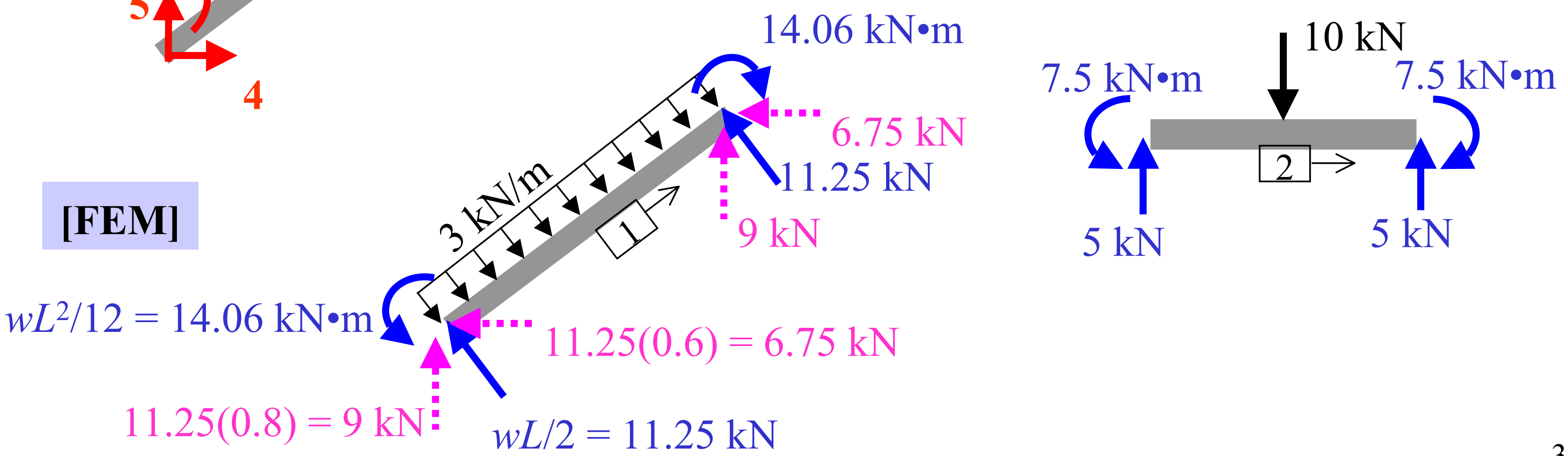
Global

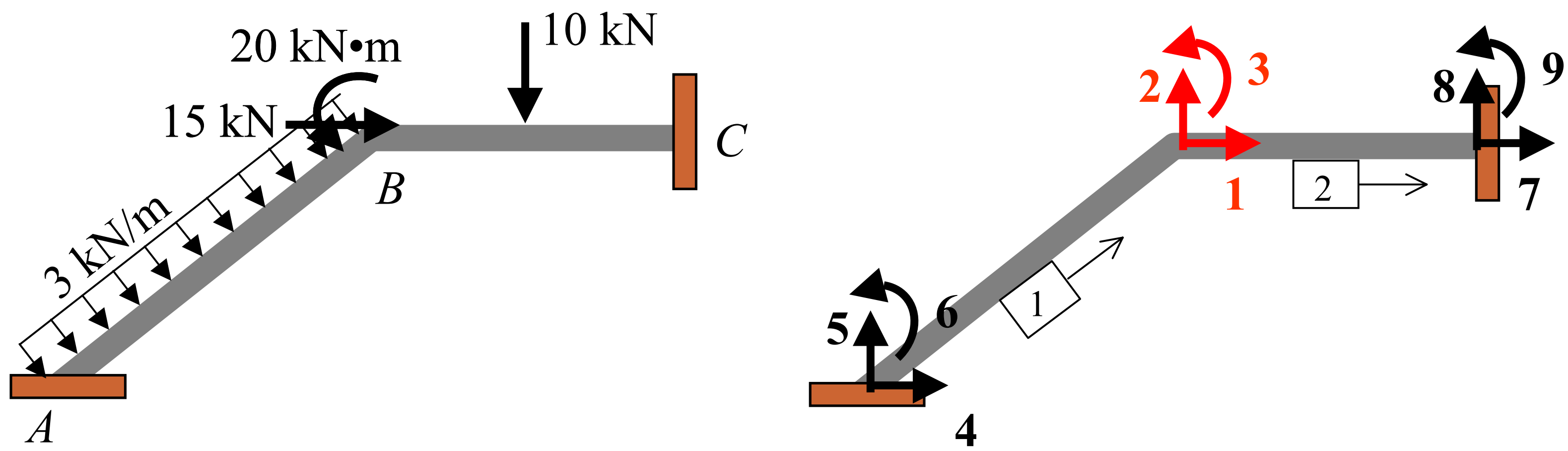


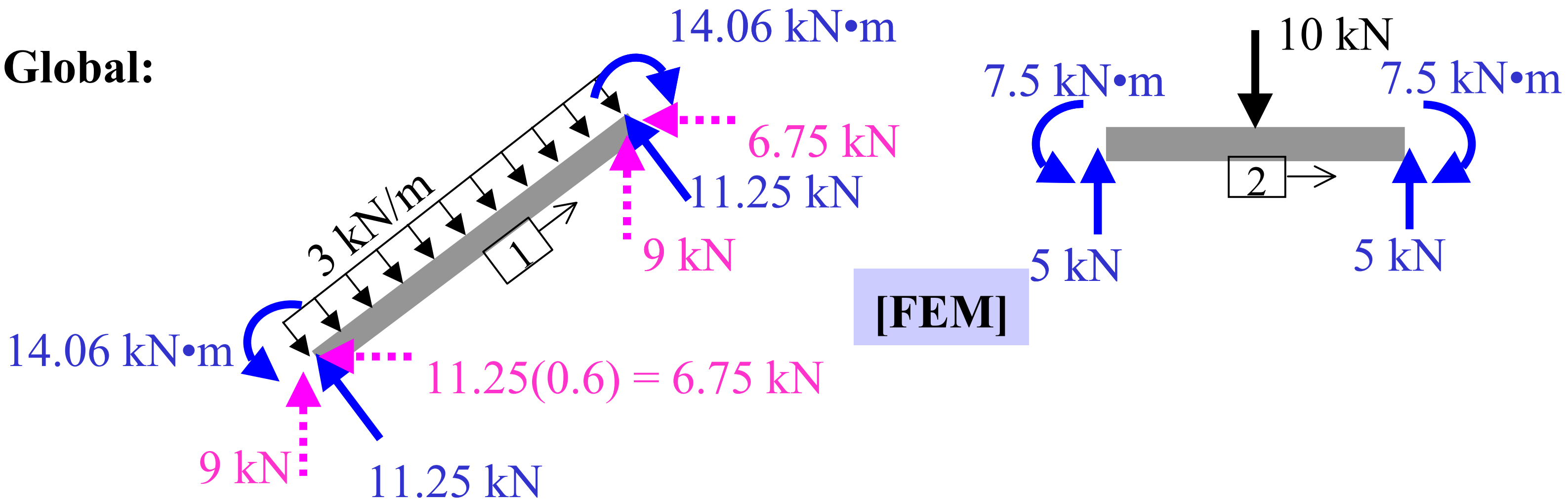
Members



[FEM]

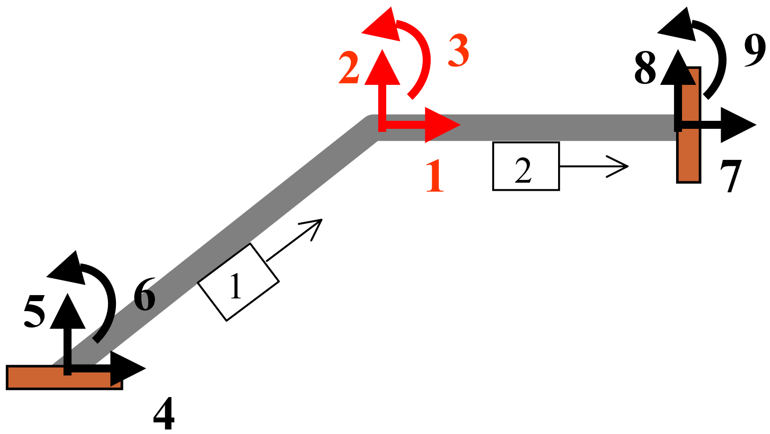






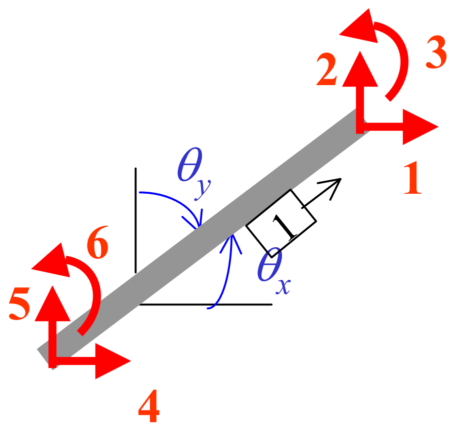
$$
\begin{array}{c} \\ \end{array}
\qquad\qquad\qquad\qquad
\begin{array}{ccc} 1 & 2 & 3 \end{array}
$$

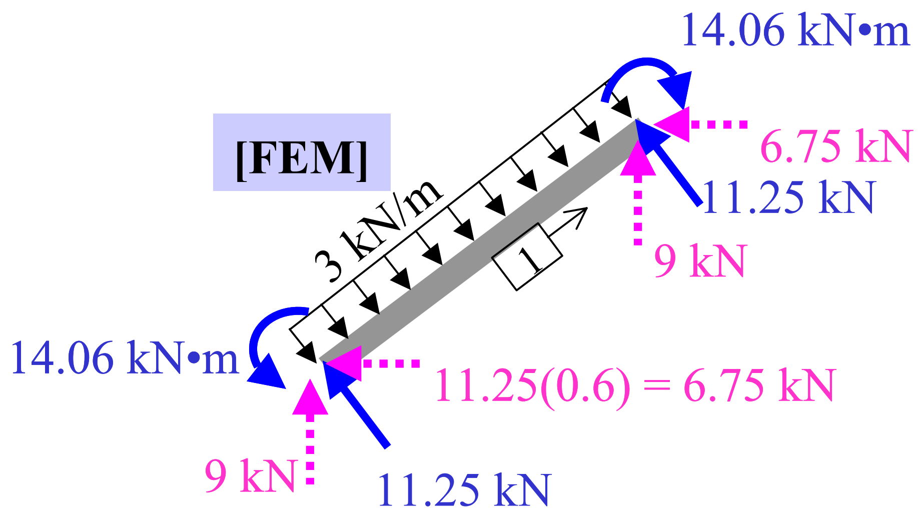
$$
\begin{bmatrix} Q_1 = 15 \\ Q_2 = 0 \\ Q_3 = 20 \end{bmatrix}
=
\begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\begin{bmatrix} 30362.9 & 7516.16 & 768 \\ 7516.16 & 6645.2 & 976 \\ 768 & 976 & 14400 \end{bmatrix}
\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}
+
\begin{bmatrix} -6.75 \\ 9 + 5 \\ -14.06 + 7.5 \end{bmatrix}
$$



$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 1.751(10^{-3})\ \text{m} \\ -4.388(10^{-3})\ \text{m} \\ 2.049(10^{-3})\ \text{rad} \end{pmatrix}$$

Member 1:

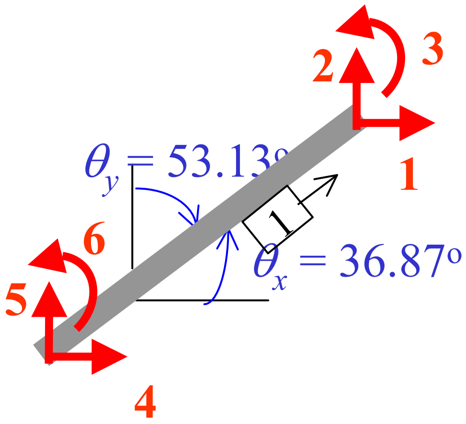




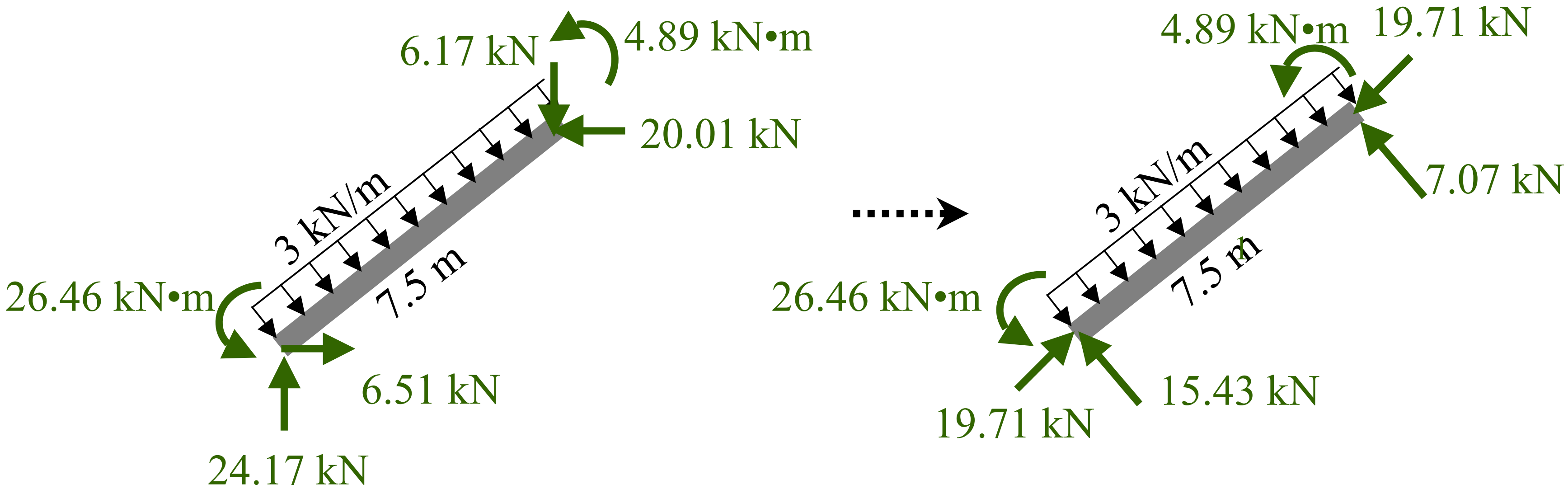
$\lambda_x = \cos 36.87^{o} = 0.8,$

$\lambda_y = \cos 53.13^{o} = 0.6$

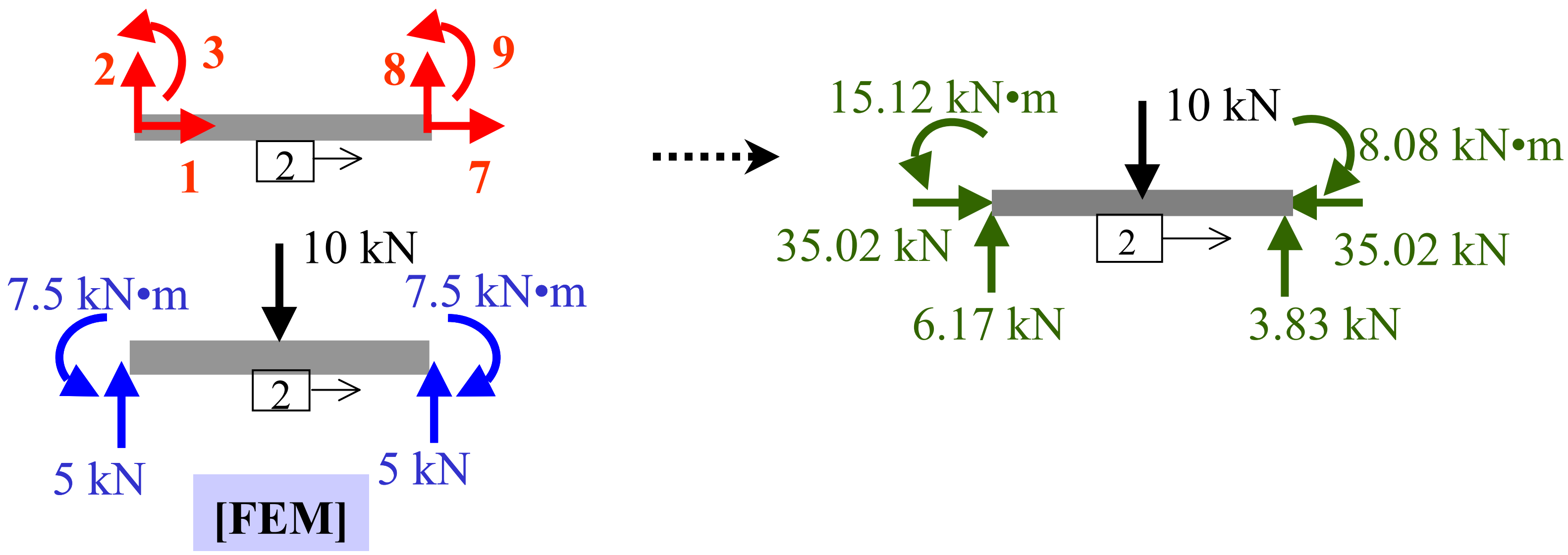
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{matrix} & \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} & & & & & \\ & & & & & \\ & & \boldsymbol{k_1} & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_1 = 1.751(10^{-3}) \\ D_2 = -4.388(10^{-3}) \\ D_3 = 2.049(10^{-3}) \end{pmatrix} + \begin{pmatrix} -6.75 \\ 9 \\ 14.06 \\ -6.75 \\ 9 \\ -14.06 \end{pmatrix} = \begin{pmatrix} 6.51 \\ 24.17 \\ 26.46 \\ -20.01 \\ -6.17 \\ 4.89 \end{pmatrix}$$



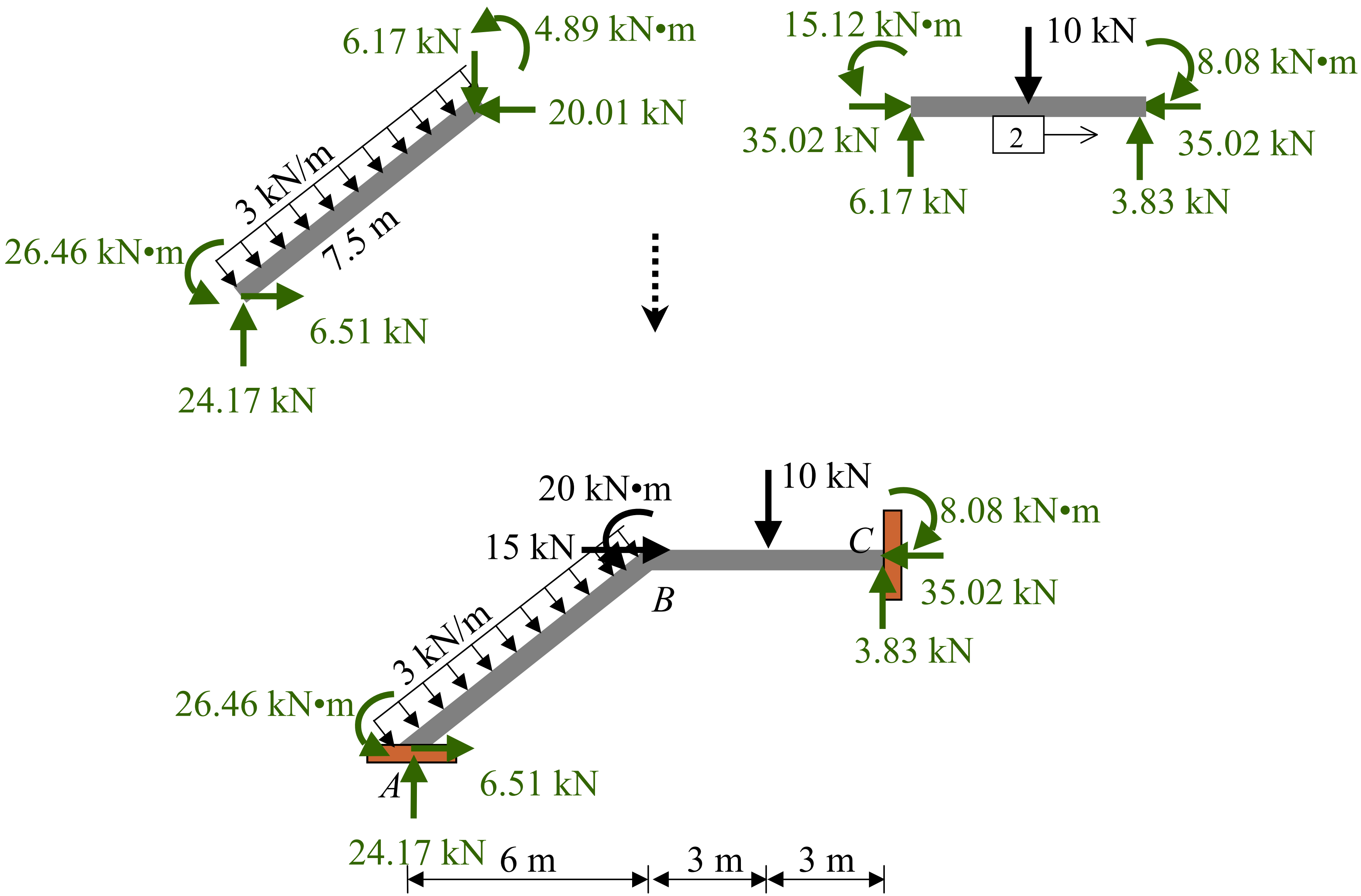
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 6.51 \\ 24.17 \\ 26.46 \\ -20.01 \\ -6.17 \\ 4.89 \end{pmatrix}$$

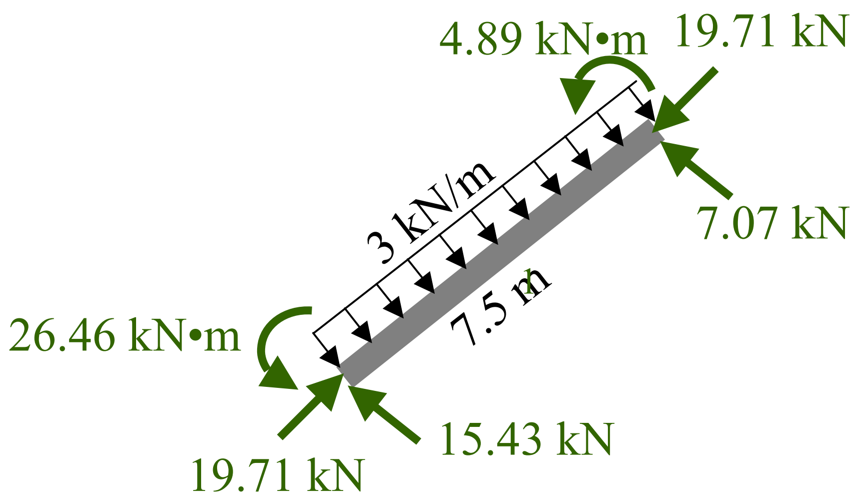


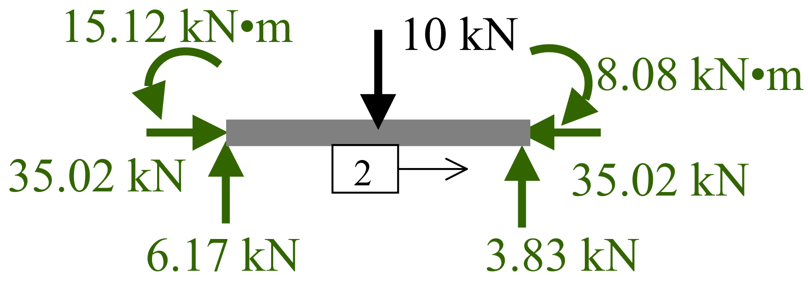
Member 2:

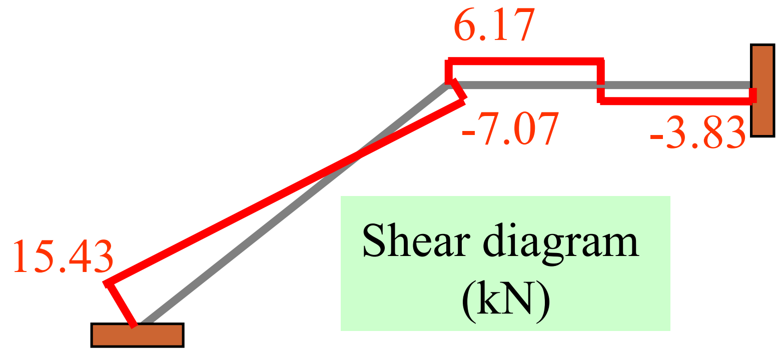


$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 7 & 8 & 9 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{matrix} & \begin{pmatrix} \\ \\ & & \boldsymbol{k_2} & & \\ \\ \\ \end{pmatrix} \end{matrix} \begin{pmatrix} D_1 = 1.751(10^{-3}) \\ D_2 = -4.388(10^{-3}) \\ D_3 = 2.049(10^{-3}) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 7.5 \\ 0 \\ 5 \\ -7.5 \end{pmatrix} = \begin{pmatrix} 35.02 \\ 6.17 \\ 15.12 \\ -35.02 \\ 3.83 \\ -8.08 \end{pmatrix}$$


6.17 kN
4.89 kN•m
20.01 kN
3 kN/m
7.5 m
26.46 kN•m
6.51 kN
24.17 kN
15.12 kN•m
10 kN
8.08 kN•m
35.02 kN
2
35.02 kN
6.17 kN
3.83 kN
20 kN•m
10 kN
8.08 kN•m
15 kN
C
B
35.02 kN
3 kN/m
3.83 kN
26.46 kN•m
A
6.51 kN
24.17 kN
6 m
3 m
3 m

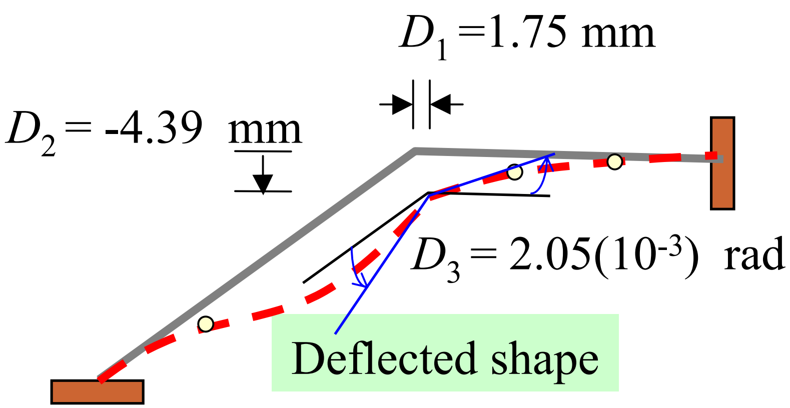




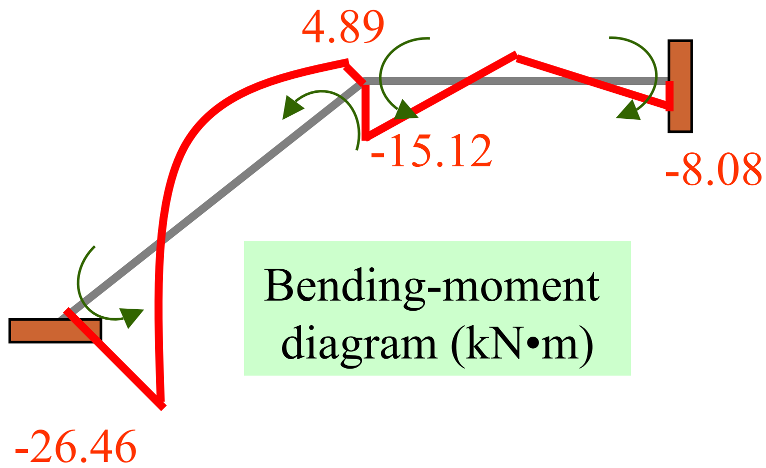


Shear diagram (kN)

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 1.751(10^{-3})\ \text{m} \\ -4.388(10^{-3})\ \text{m} \\ 2.049(10^{-3})\ \text{rad} \end{pmatrix}$$



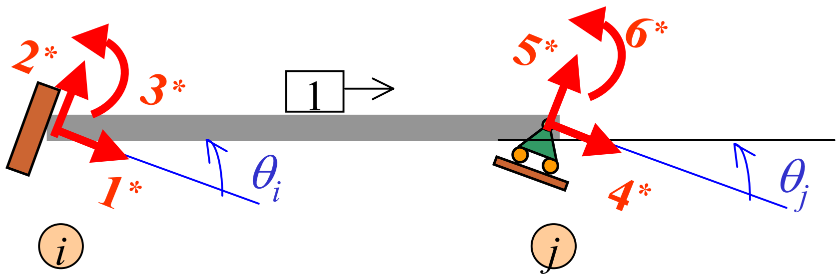
Deflected shape



Bending-moment diagram (kN•m)

Special Frames

Stiffness matrix



$\lambda_{ix} = \cos\,\theta_i$ $\qquad$ $\lambda_{jx} = \cos\,\theta_j$

$\lambda_{iy} = \sin\,\theta_i$ $\qquad$ $\lambda_{jy} = \sin\,\theta_j$

$$[\,q^*\,] = [\,T\,]^{\mathrm{T}}[\,q'\,]$$

$$\begin{bmatrix} q_{1^*} \\ q_{2^*} \\ q_{3^*} \\ q_{4^*} \\ q_{5^*} \\ q_{6^*} \end{bmatrix} = \begin{matrix} 1^* \\ 2^* \\ 3^* \\ 4^* \\ 5^* \\ 6^* \end{matrix} \overset{\begin{matrix} 1' & 2' & 3' & 4' & 5' & 6' \end{matrix}}{\begin{bmatrix} \lambda_{ix} & -\lambda_{iy} & 0 & 0 & 0 & 0 \\ \lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{jx} & -\lambda_{jy} & 0 \\ 0 & 0 & 0 & \lambda_{jy} & \lambda_{jx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$[\,T\,]^{\mathrm{T}}$

$$[T] = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1^* & 2^* & 3^* & 4^* & 5^* & 6^* \end{array} \\ \begin{bmatrix} \lambda_{ix} & \lambda_{iy} & 0 & 0 & 0 & 0 \\ -\lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{jx} & \lambda_{jy} & 0 \\ 0 & 0 & 0 & -\lambda_{jy} & \lambda_{jx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- **Member Stiffness Matrix**

$$[k'] = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

$$[\,\boldsymbol{k}\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{k}'\,][\boldsymbol{T}] =$$

$$
\begin{array}{c}
 \\ U_i \\ V_i \\ M_i \\ U_j \\ V_j \\ M_j
\end{array}
\begin{array}{cccccc}
U_i & V_i & M_i & U_j & V_j & M_j \\
\left(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & -\frac{6EI}{L^2}\lambda_{iy} & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{iy} \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & \left(\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2\right) & \frac{6EI}{L^2}\lambda_{ix} & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & \frac{6EI}{L^2}\lambda_{ix} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{2EI}{L} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & \frac{6EI}{L^2}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \frac{6EI}{L^2}\lambda_{jy} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{jx} & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2\right) & -\frac{6EI}{L^2}\lambda_{jx} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{4EI}{L}
\end{array}
$$

Example 4

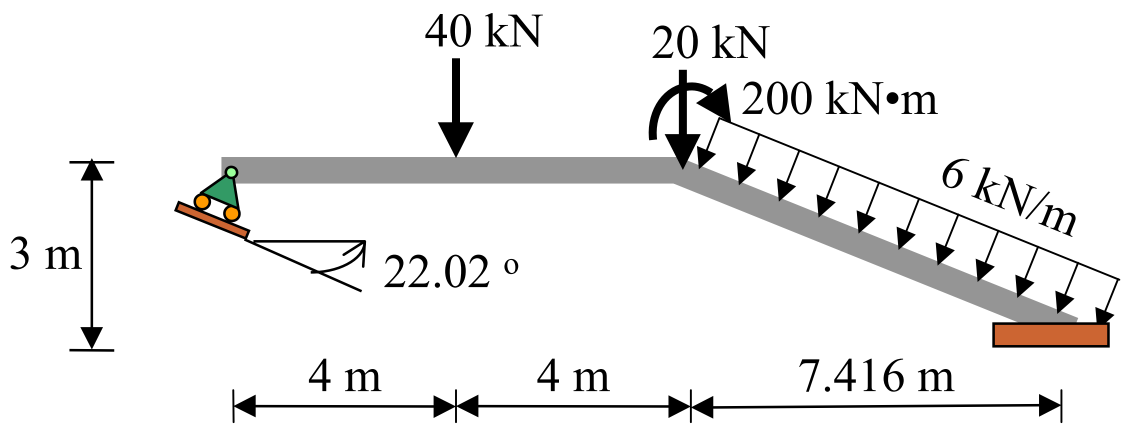
For the beam shown:

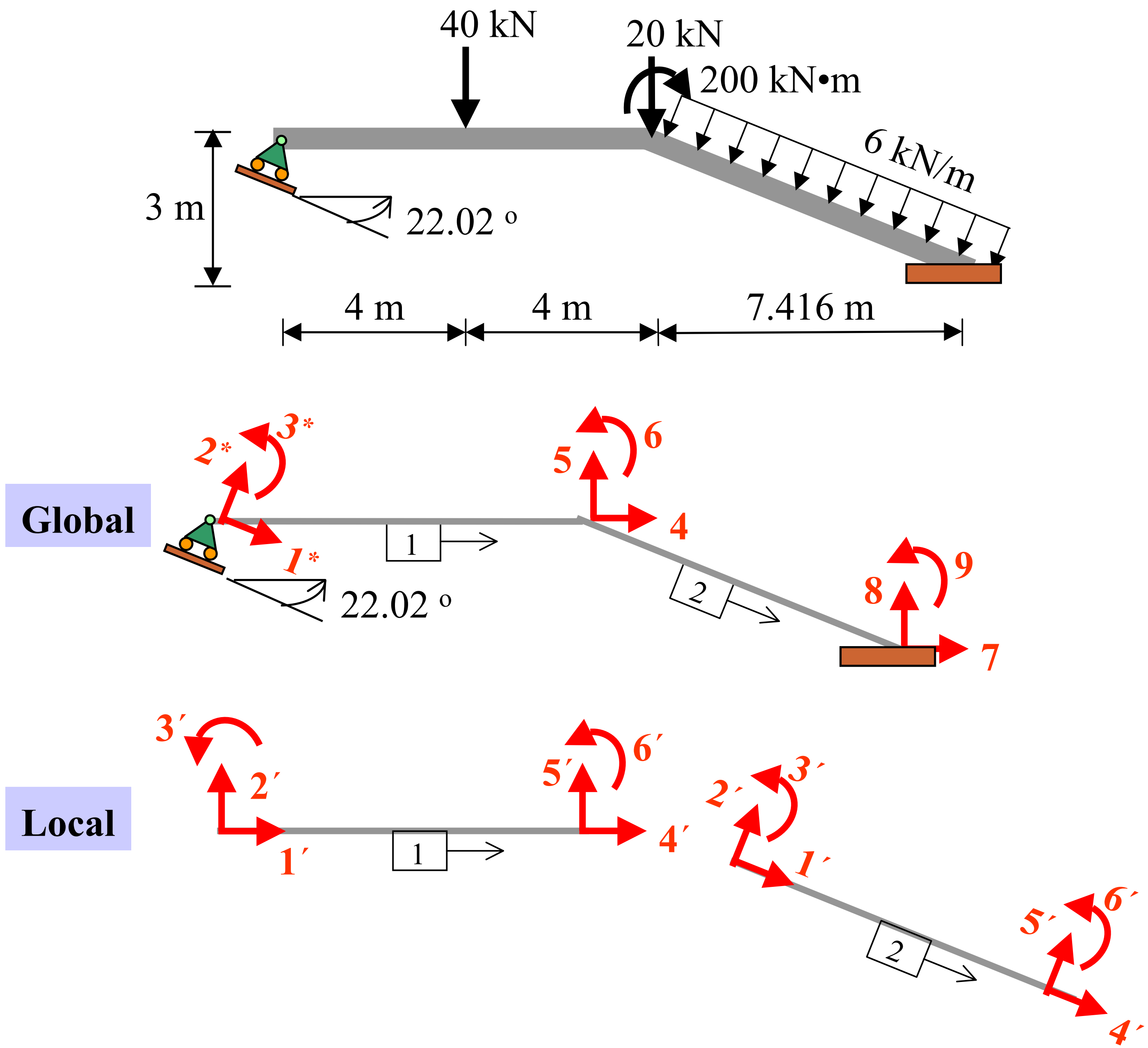
(a) Use the stiffness method to determine all the **reactions** at supports.

(b) Draw the **quantitative free-body diagram** of member.

(c) Draw the **quantitative bending moment diagrams** and **qualitative deflected shape.**

Take $I = 200(10^6)$ mm^4 , $A = 6(10^3)$ mm^2, and $E = 200$ GPa for all members. Include axial deformation in the stiffness matrix.




40 kN
20 kN
200 kN•m
6 kN/m
3 m
22.02 o
4 m
4 m
7.416 m
Global
Local

$$
[\mathrm{k'}] =
\begin{array}{c}
 \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

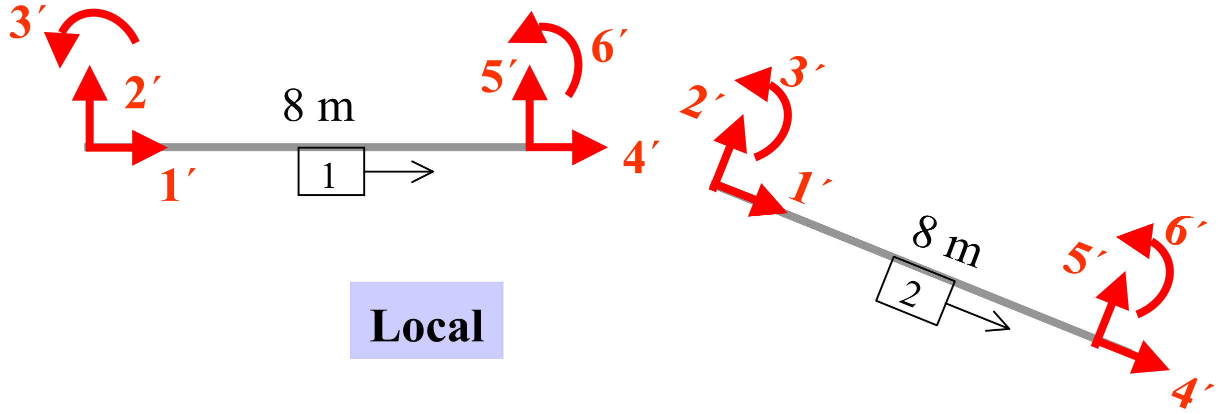
$$\frac{AE}{L} = \frac{(0.006\,m^2)(200\times10^6\,kN/m^2)}{8\,m} = 150\times10^3\ kN/m$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\,kN/m^2)(0.0002\,m^4)}{8\,m} = 20\times10^3\ kN\bullet m$$

$$\frac{2EI}{L} = 10\times10^3\ kN\bullet m$$

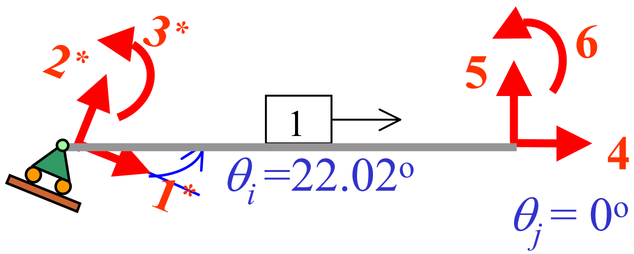
$$\frac{6EI}{L^2} = \frac{6(200\times10^6\,kN/m^2)(0.0002\,m^4)}{(8\,m)^2} = 3.75\times10^3\ kN$$

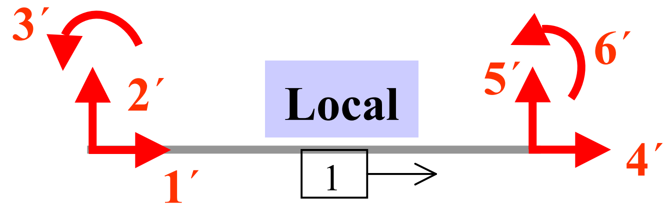
$$\frac{12EI}{L^3} = \frac{12(200\times10^6\,kN/m^2)(0.0002\,m^4)}{(8\,m)^3} = 0.9375\times10^3\ kN/m$$



$$[\,k'\,]_1 = [\,k'\,]_2 = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\ \begin{pmatrix} 150000 & 0 & 0 & -150000 & 0 & 0 \\ 0 & 937.5 & 3750 & 0 & -937.5 & 3750 \\ 0 & 3750 & 20000 & 0 & -3750 & 10000 \\ -150000 & 0 & 0 & 150000 & 0 & 0 \\ 0 & -937.5 & -3750 & 0 & 937.5 & -3750 \\ 0 & 3750 & 10000 & 0 & -3750 & 20000 \end{pmatrix} \end{array}$$

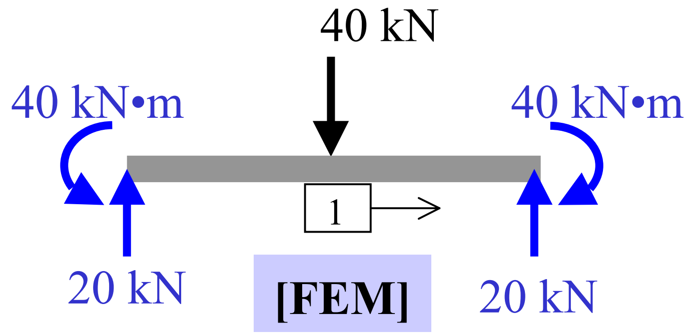
Member 1:





λ_{ix} = cos (22.02°) = 0.927, λ_{jx} = cos (0°) = 1,

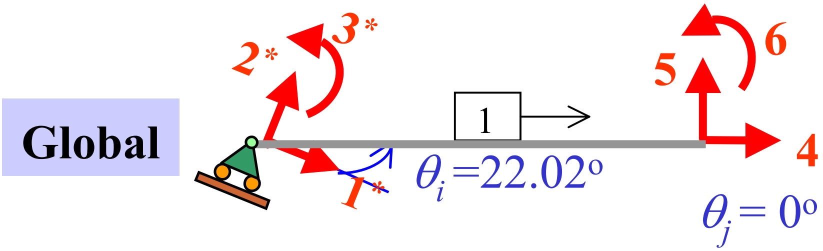
λ_{iy} = sin (22.02°) = 0.375 λ_{jy} = sin (0°) = 0

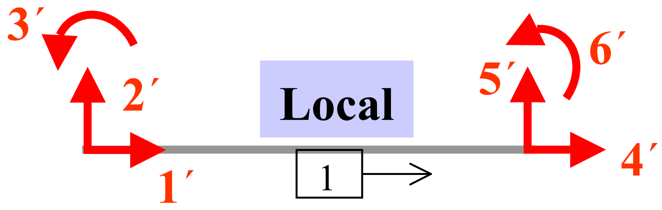


$$[\, q^* \,] = [\, T \,]^{T} [\, q' \,]$$

$$\begin{pmatrix} q^*_1 \\ q^*_2 \\ q^*_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{array}{c|cccccc} & 1' & 2' & 3' & 4' & 5' & 6' \\ \hline 1^* & 0.927 & -0.375 & 0 & 0 & 0 & 0 \\ 2^* & 0.375 & 0.927 & 0 & 0 & 0 & 0 \\ 3^* & 0 & 0 & 1 & 0 & 0 & 0 \\ 4^* & 0 & 0 & 0 & 1 & 0 & 0 \\ 5^* & 0 & 0 & 0 & 0 & 1 & 0 \\ 6^* & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \begin{pmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{pmatrix}$$

$[\, T \,]_1^{T}$

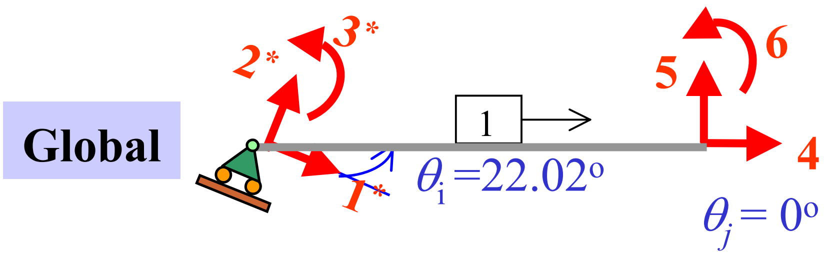


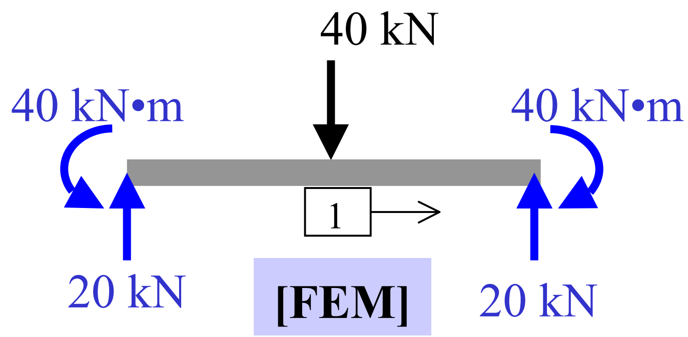


$$[\,\boldsymbol{k}^*\,]_1 = [\,\boldsymbol{T}\,]_1^{\mathrm{T}}[\,\boldsymbol{k}'\,]_1[\,\boldsymbol{T}\,]_1$$

$$[\,k^*\,]_1 = 10^3$$

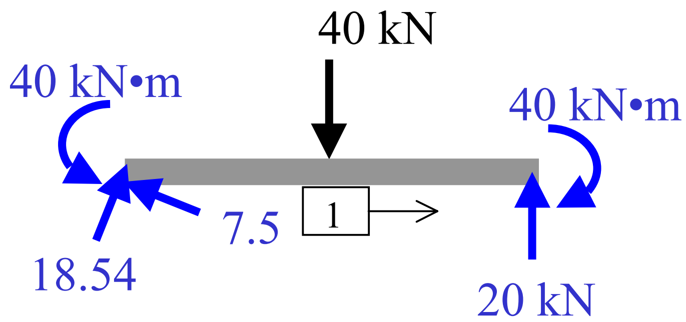
	1*	2*	3*	4	5	6
1*	129.046	51.811	-1.406	-139.058	0.351	-1.406
2*	51.811	21.892	3.476	-56.240	-0.869	3.476
3*	-1.406	3.476	20.00	0	-3.75	10.00
4	-139.058	-56.240	0.00	150	0	0
5	0.351	-0.869	-3.75	0	0.938	-3.75
6	-1.406	3.476	10.00	0	-3.75	20



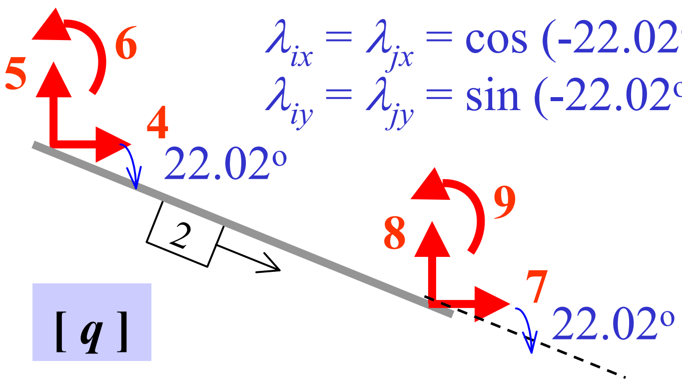


$$[\,q^{F*}\,] = [\,T\,]^T[\,q^{F\prime}]$$

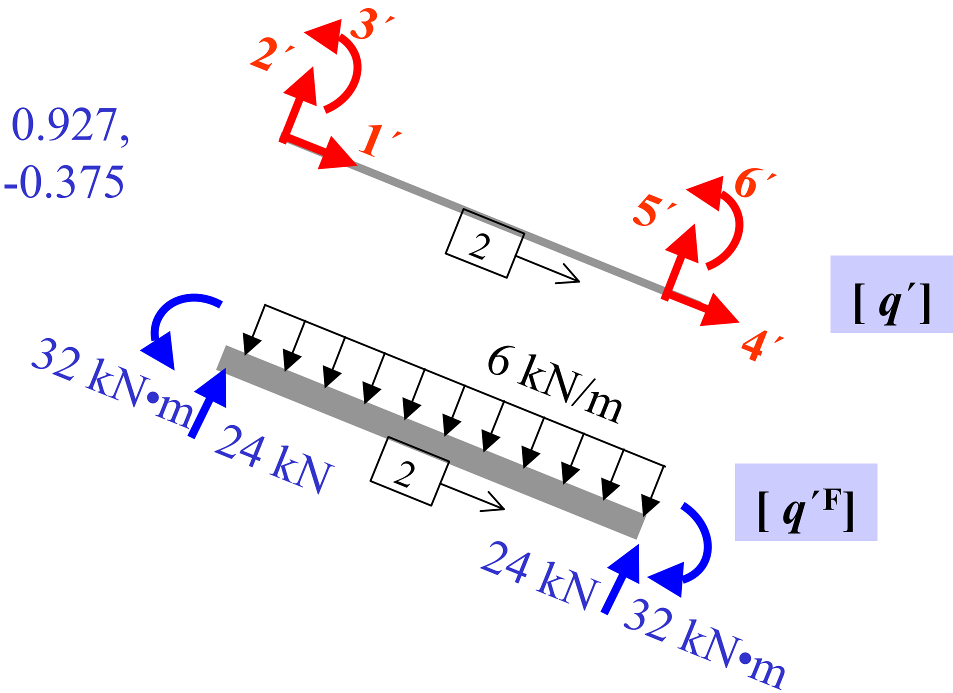
$$[\,q^{F*}] = [\,T\,]_1^{\,T}\begin{pmatrix} 0 \\ 20 \\ 40 \\ 0 \\ 20 \\ -40 \end{pmatrix} = \begin{pmatrix} -7.50 \\ 18.54 \\ 40 \\ 0 \\ 20 \\ -40 \end{pmatrix}\begin{matrix} 1^* \\ 2^* \\ 3^* \\ 4 \\ 5 \\ 6 \end{matrix}$$



Member 2



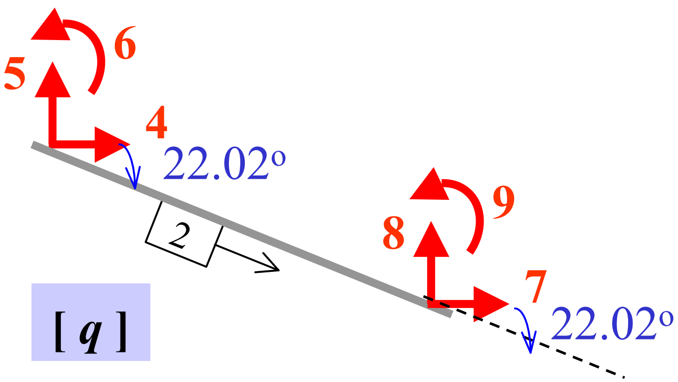
$$\lambda_{ix} = \lambda_{jx} = \cos(-22.02^{o}) = 0.927,$$
$$\lambda_{iy} = \lambda_{jy} = \sin(-22.02^{o}) = -0.375$$

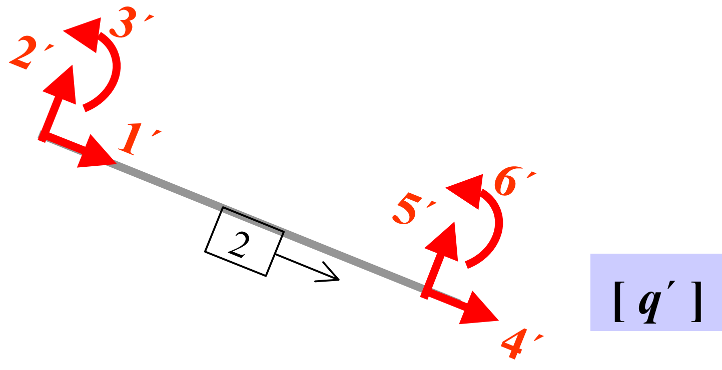


$$[\,q\,] = [\,T\,]^{T}[\,q'\,]$$

$$\begin{Bmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{Bmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \\ 0.927 & 0.375 & 0 & 0 & 0 & 0 \\ -0.375 & 0.927 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.927 & 0.375 & 0 \\ 0 & 0 & 0 & -0.375 & 0.927 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \begin{Bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{Bmatrix}$$

$[\,T\,]_2^{T}$

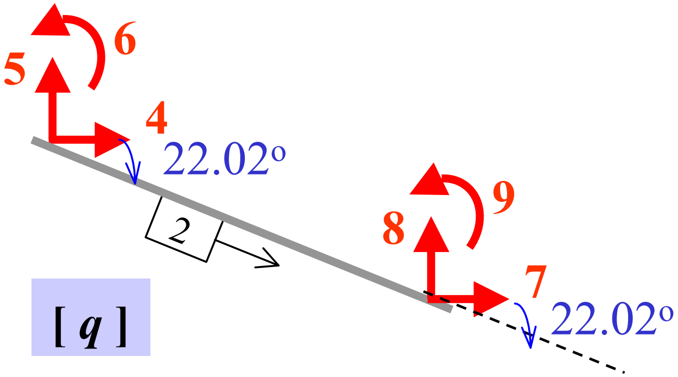


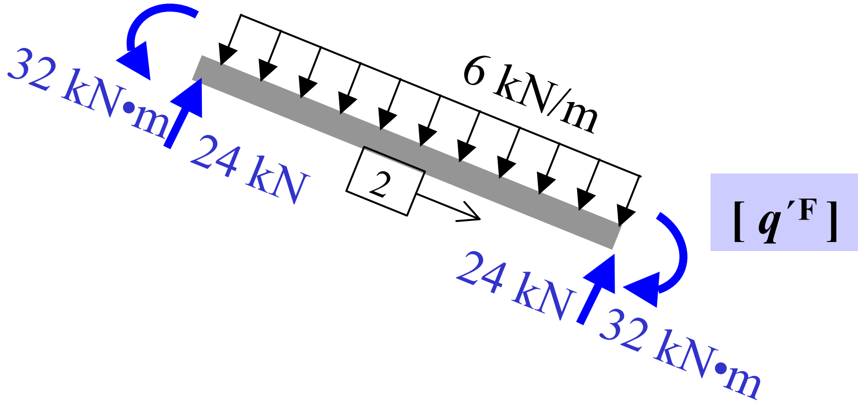


$$[\,\boldsymbol{k}\,]_2 = [\,\boldsymbol{T}\,]_2^{\mathrm{T}}[\,\boldsymbol{k}'\,]_2[\,\boldsymbol{T}\,]_2$$

$[\,k\,]_2 = 10^3$

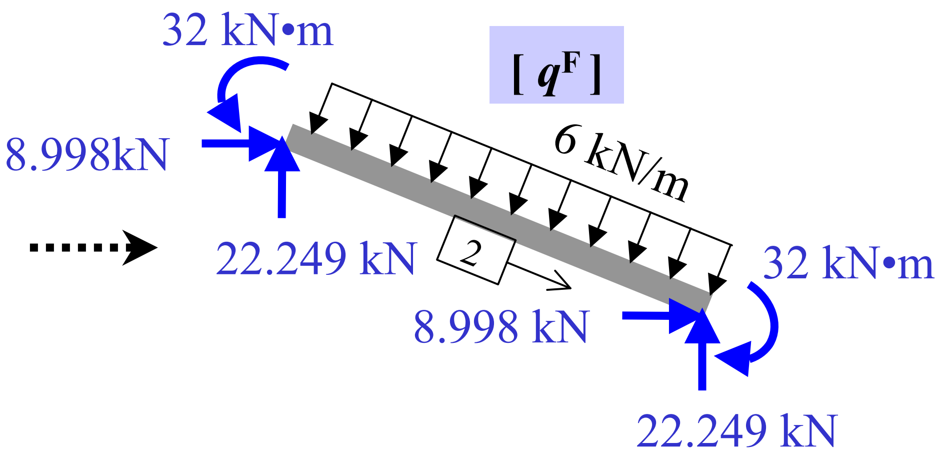
	4	5	6	7	8	9
4	129.046	-51.811	1.406	-129.046	51.811	1.406
5	-51.811	21.892	3.476	51.811	-21.892	3.476
6	1.406	3.476	20	-1.406	-3.476	10
7	-129.046	51.811	-1.406	129.046	-51.811	-1.406
8	51.811	-21.892	-3.476	-51.811	21.892	-3.476
9	1.4056	3.476	10	-1.406	-3.476	20



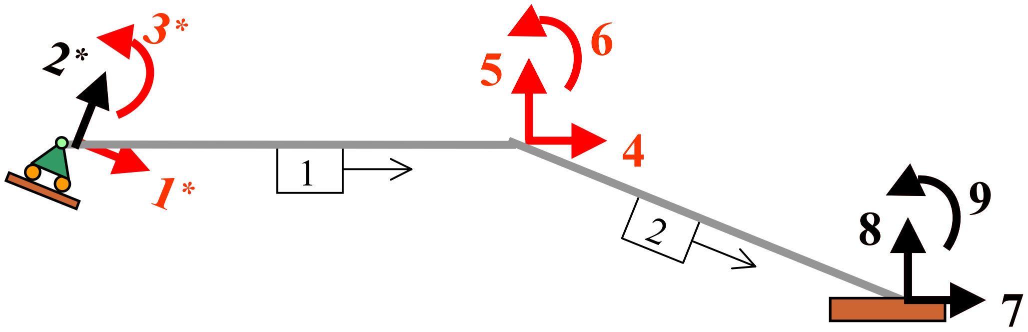


$$[\,q^{F*}\,] = [\,T\,]^{T}[\,q^{F\prime}\,]$$

$$[\,q^{F}\,] = [\,T\,]_2^{T}\begin{pmatrix} 0 \\ 24 \\ 32 \\ 0 \\ 24 \\ -32 \end{pmatrix} = \begin{pmatrix} 8.998 \\ 22.249 \\ 32 \\ 8.998 \\ 22.249 \\ -32 \end{pmatrix}\begin{matrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix}$$



Global Stiffness:

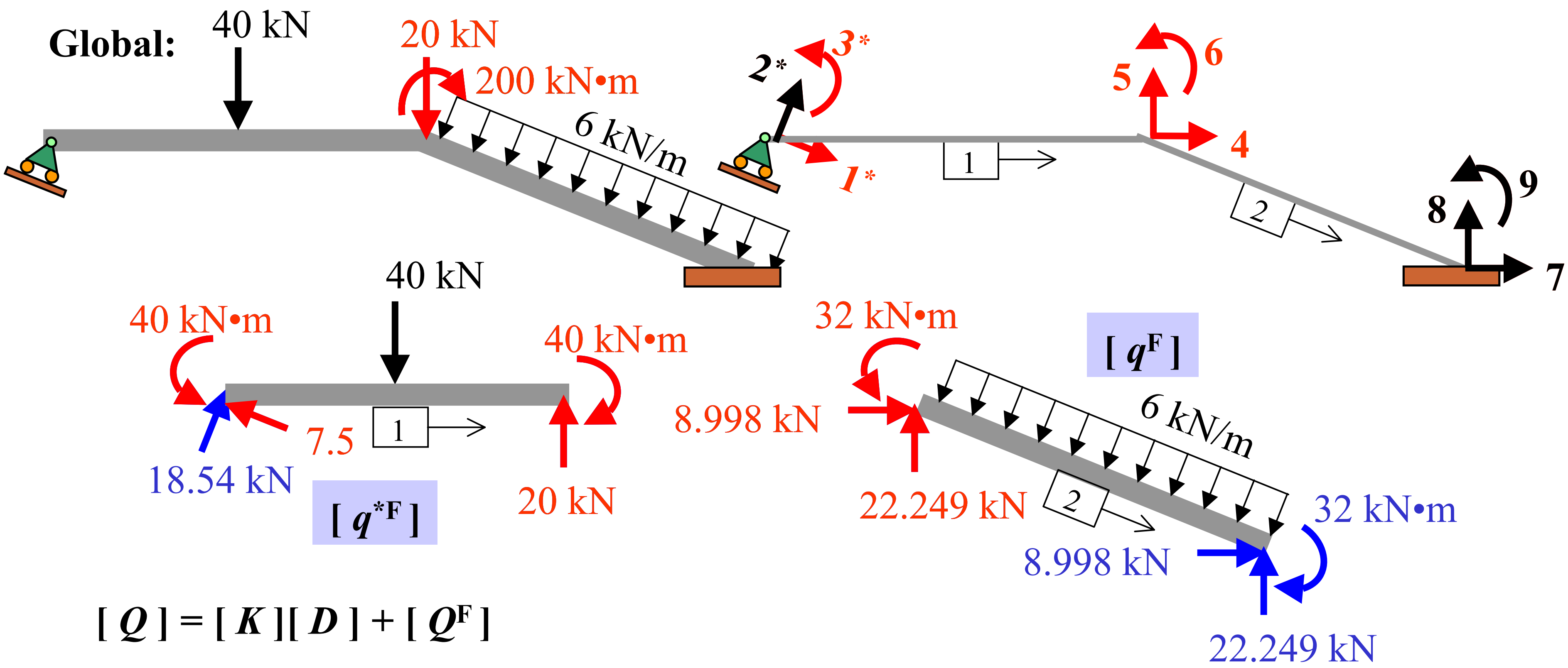


$[k^*]_1 = 10^3$

	1*	2*	3*	4	5	6
1*	129.046	51.811	-1.406	-139.058	0.351	-1.406
2*	51.811	21.892	3.476	-56.240	-0.869	3.476
3*	-1.406	3.476	20.00	0	-3.75	10.00
4	-139.058	-56.240	0.00	150	0	0
5	0.351	-0.869	-3.75	0	0.938	-3.75
6	-1.406	3.476	10.00	0	-3.75	20

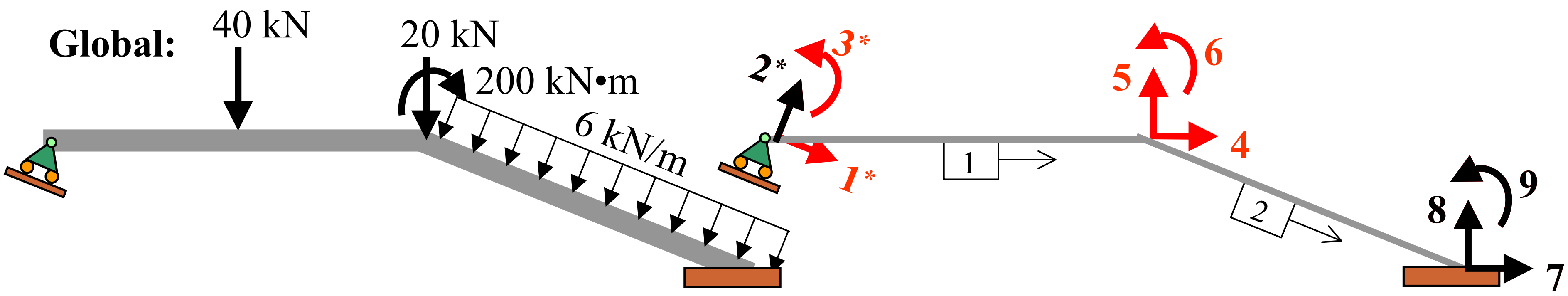
$[k]_2 = 10^3$

	4	5	6	7	8	9
4	129.046	-51.811	1.406	-129.046	51.811	1.406
5	-51.811	21.892	3.476	51.811	-21.892	3.476
6	1.406	3.476	20	-1.406	-3.476	10
7	-129.046	51.811	-1.406	129.046	-51.811	-1.406
8	51.811	-21.892	-3.476	-51.811	21.892	-3.476
9	1.4056	3.476	10	-1.406	-3.476	20

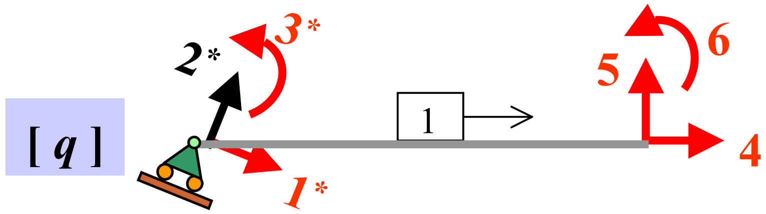


$[\,Q\,] = [\,K\,][\,D\,] + [\,Q^F\,]$

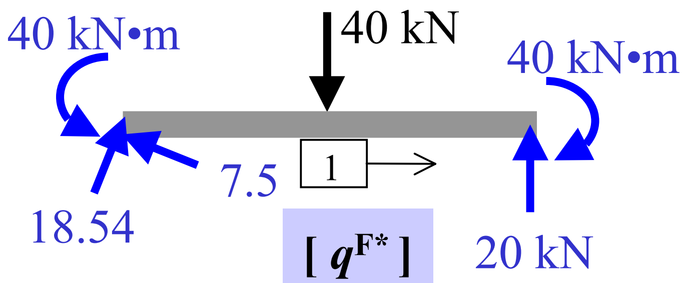
$$
\begin{pmatrix} Q_{1*} = 0.0 \\ Q_{3*} = 0.0 \\ Q_4 = 0.0 \\ Q_5 = -20 \\ Q_6 = -200 \end{pmatrix}
= 10^3
\begin{array}{c} \\ 1* \\ 3* \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{ccccc} 1* & 3* & 4 & 5 & 6 \\ 129.046 & -1.406 & -139.058 & 0.351 & -1.406 \\ -1.406 & 20 & 0 & -3.75 & 10 \\ -139.058 & 0 & 279.046 & -51.811 & 1.406 \\ 0.352 & -3.75 & -51.811 & 22.829 & -0.274 \\ -1.406 & 10 & 1.406 & -0.274 & 40 \end{array}
\begin{pmatrix} D_{1*} \\ D_{3*} \\ D_4 \\ D_5 \\ D_6 \end{pmatrix}
+
\begin{pmatrix} -7.5 \\ 40 \\ 8.998 \\ 20 + 22.249 \\ -40 + 32 \end{pmatrix}
$$



$$\begin{pmatrix} D_{1*} \\ D_{3*} \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} = \begin{pmatrix} -0.0205 & \text{m} \\ -0.0112 & \text{rad} \\ -0.0191 & \text{m} \\ -0.0476 & \text{m} \\ -0.0024 & \text{rad} \end{pmatrix}$$

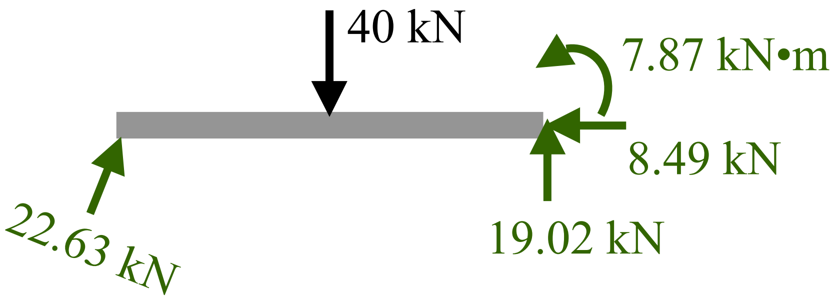


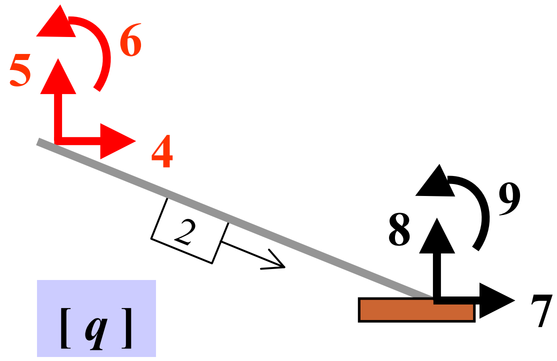
Member 1: [q*] = [k*][d*] + [qF*]



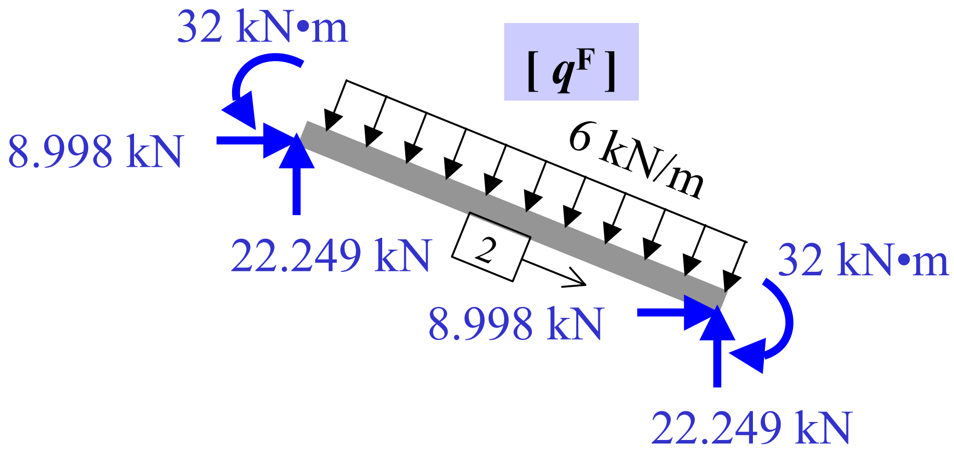
$$\begin{bmatrix} q_{1*} \\ q_{2*} \\ q_{3*} \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = 10^3 \begin{array}{c} \\ 1* \\ 2* \\ 3* \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1* & 2* & 3* & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 129.046 & 51.811 & -1.406 & -139.058 & 0.351 & -1.406 \\ 51.811 & 21.892 & 3.476 & -56.240 & -0.869 & 3.476 \\ -1.406 & 3.476 & 20.00 & 0 & -3.75 & 10.00 \\ -139.058 & -56.240 & 0.00 & 150 & 0 & 0 \\ 0.351 & -0.869 & -3.75 & 0 & 0.938 & -3.75 \\ -1.406 & 3.476 & 10.00 & 0 & -3.75 & 20 \end{bmatrix} \end{array} \begin{bmatrix} D_{1*}=-0.0205 \\ D_{2*}=0.0 \\ D_{3*}=-0.0112 \\ D_4=-0.0191 \\ D_5=-0.0476 \\ D_6=-0.0024 \end{bmatrix} + \begin{bmatrix} -7.50 \\ 18.54 \\ 40 \\ 0 \\ 20 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} q_{1*} \\ q_{2*} \\ q_{3*} \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 & \\ 22.63 & \text{kN} \\ 0 & \\ -8.49 & \text{kN} \\ 19.02 & \text{kN} \\ 7.87 & \text{kN}\cdot\text{m} \end{bmatrix}$$



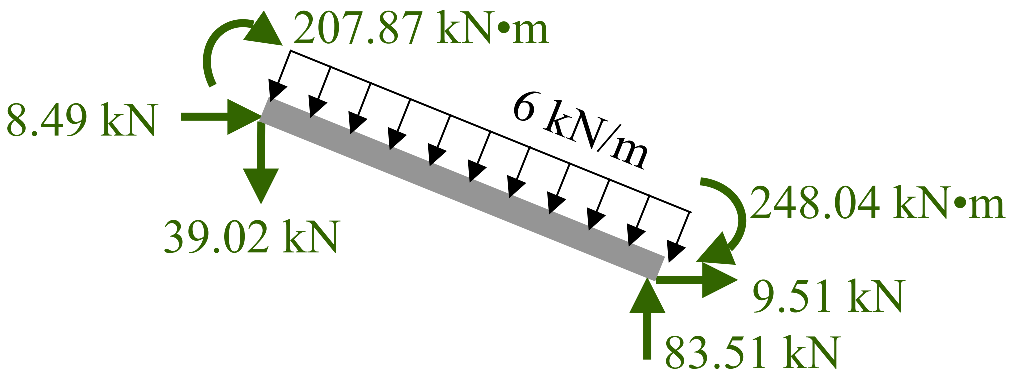


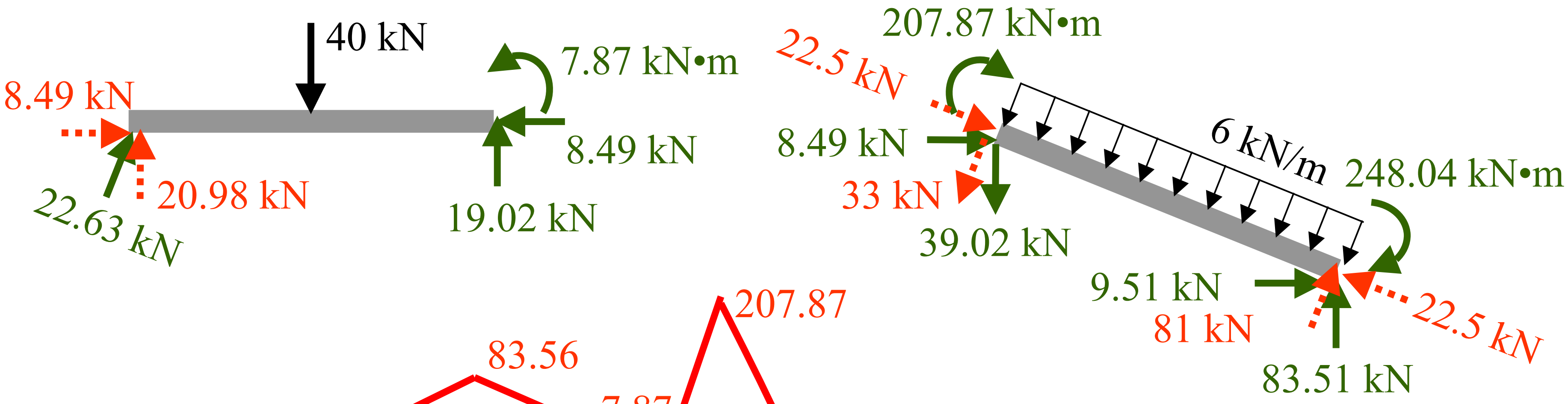
Member 2 : [q] = [k][d] + [qF]

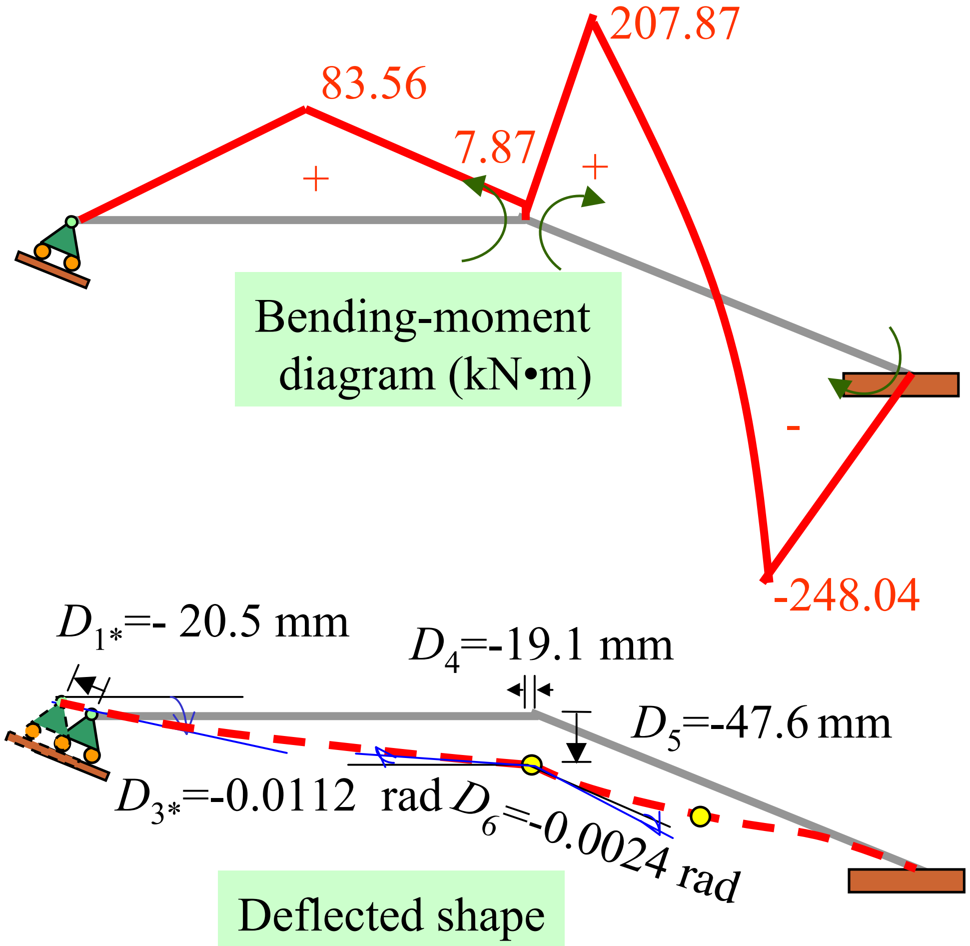


$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 129.046 & -51.811 & 1.406 & -129.046 & 51.811 & 1.406 \\ -51.811 & 21.892 & 3.476 & 51.811 & -21.892 & 3.476 \\ 1.406 & 3.476 & 20 & -1.406 & -3.476 & 10 \\ -129.046 & 51.811 & -1.406 & 129.046 & -51.811 & -1.406 \\ 51.811 & -21.892 & -3.476 & -51.811 & 21.892 & -3.476 \\ 1.4056 & 3.476 & 10 & -1.406 & -3.476 & 20 \end{pmatrix} \end{array} \begin{pmatrix} D_4 = -0.0191 \\ D_5 = -0.0476 \\ D_6 = -0.0024 \\ D_7 = 0 \\ D_8 = 0 \\ D_9 = 0 \end{pmatrix} + \begin{pmatrix} 8.998 \\ 22.249 \\ 32 \\ 8.998 \\ 22.249 \\ -32 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{pmatrix} 8.49 & \text{kN} \\ -39.02 & \text{kN} \\ -207.87 & \text{kN•m} \\ 9.51 & \text{kN} \\ 83.51 & \text{kN} \\ -248.04 & \text{kN•m} \end{pmatrix}$$



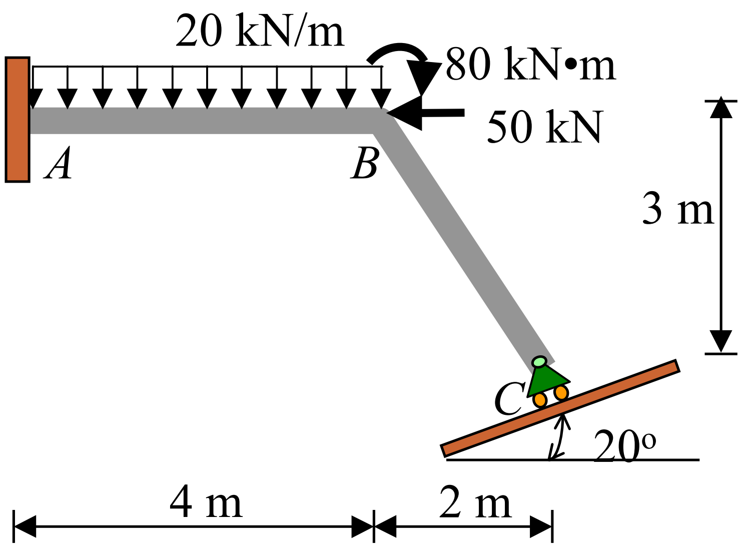


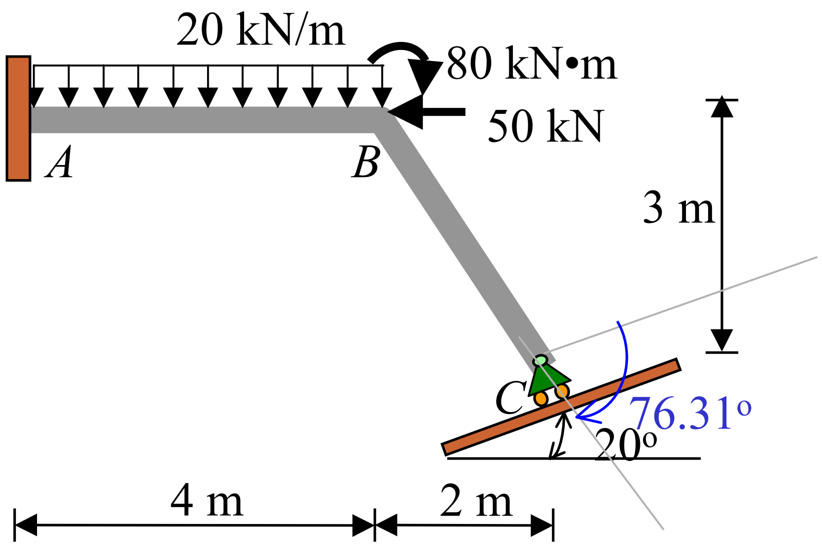


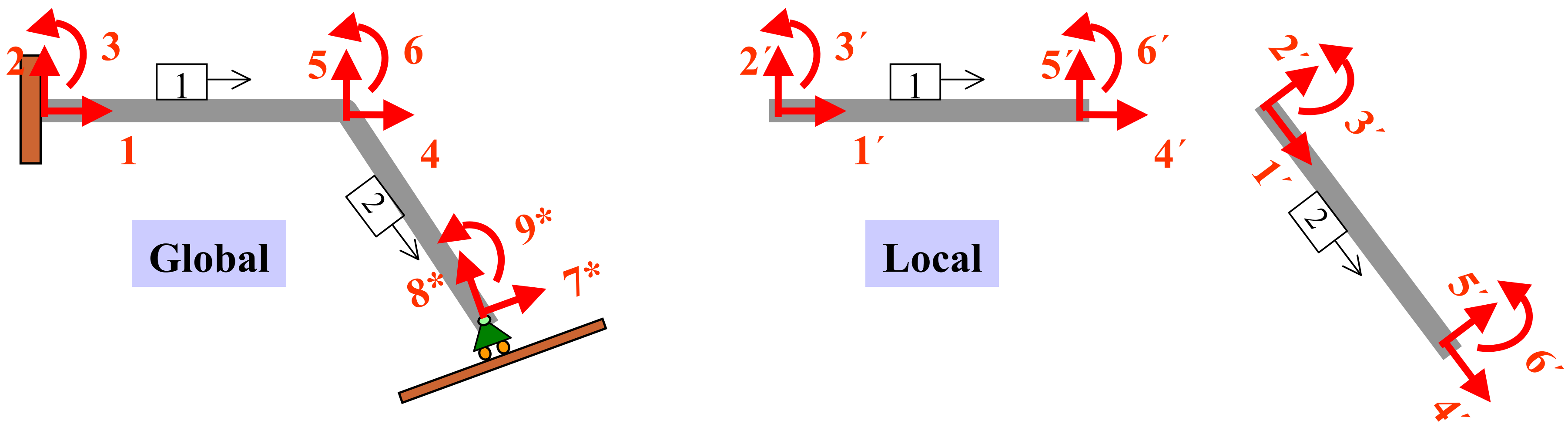
$$\begin{pmatrix} D_{1*} \\ D_{3*} \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} = \begin{pmatrix} -0.0205 & \text{m} \\ -0.0112 & \text{rad} \\ -0.0191 & \text{m} \\ -0.0476 & \text{m} \\ -0.0024 & \text{rad} \end{pmatrix}$$

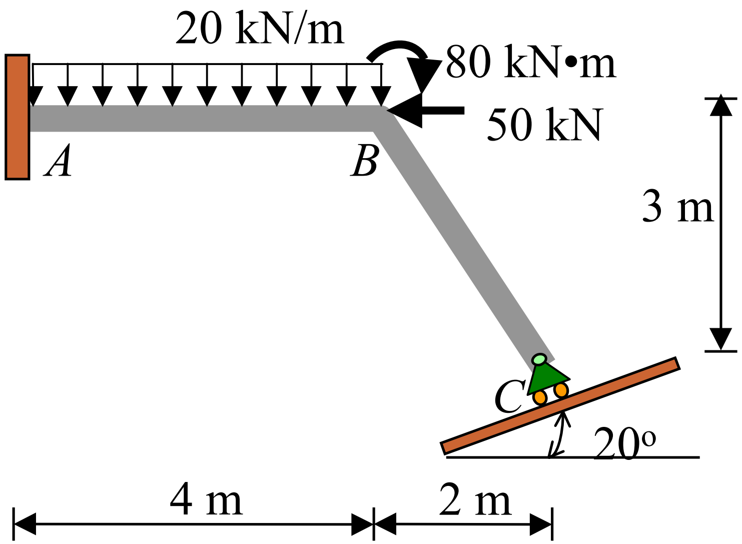
Example 5

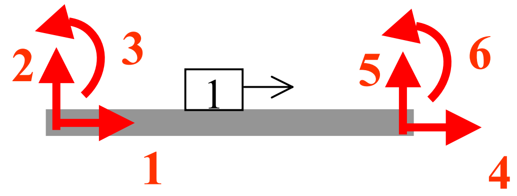
For the beam shown:
(a) Use the stiffness method to determine all the **reactions** at supports.
(b) Draw the **quantitative free-body diagram** of member.
(c) Draw the **quantitative bending moment diagrams**
and **qualitative deflected shape.**
Take $I = 400(10^6)$ mm^4 , $A = 60(10^3)$ mm^2, and E = 200 GPa for all members.




20 kN/m
80 kN•m
50 kN
A
B
C
3 m
76.31°
20°
4 m
2 m


1
2
3
4
5
6
7*
8*
9*
Global
Local
1′
2′
3′
4′
5′
6′





Member 1:

$$\frac{AE}{L} = \frac{(60 \times 10^{-3}\ \text{m}^2)(200 \times 10^6\ \text{kN/m}^2)}{4\ \text{m}}$$

$$= 3000 \times 10^3\ \text{kN/m}$$

$$\frac{4EI}{L} = \frac{4(200 \times 10^6\ \text{kN/m}^2)(400 \times 10^{-6}\text{m}^4)}{4\ \text{m}}$$

$$= 80 \times 10^3\ \text{kN} \bullet \text{m}$$

$$\frac{2EI}{L} = \frac{2(200 \times 10^6\ \text{kN/m}^2)(400 \times 10^{-6}\text{m}^4)}{4\ \text{m}}$$

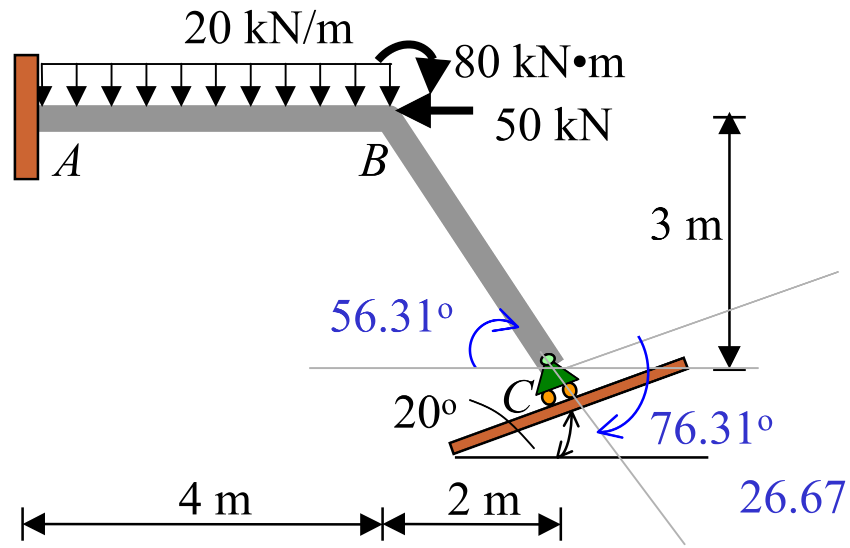
$$= 40 \times 10^3\ \text{kN} \bullet \text{m}$$

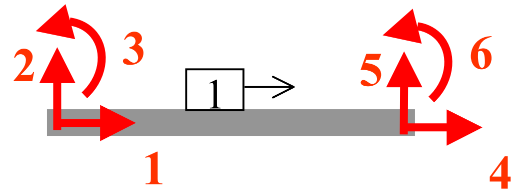
$$\frac{6EI}{L^2} = \frac{6(200 \times 10^6\ \text{kN/m}^2)(400 \times 10^{-6}\text{m}^4)}{(4\ \text{m})^2}$$

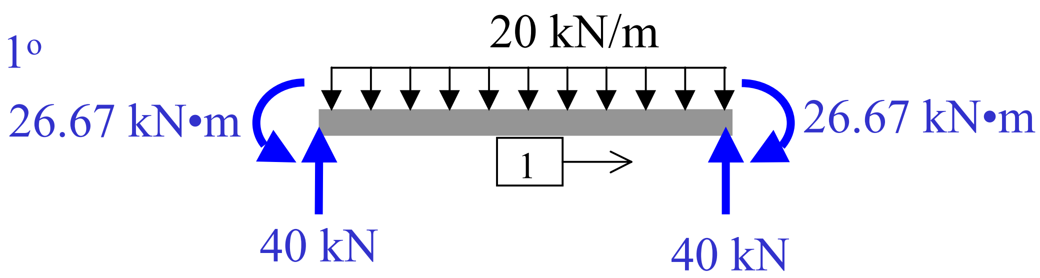
$$= 30 \times 10^3\ \text{kN}$$

$$\frac{12EI}{L^3} = \frac{12(200 \times 10^6\ \text{kN/m}^2)(400 \times 10^{-6}\text{m}^4)}{(4\ \text{m})^3}$$

$$= 15 \times 10^3\ \text{kN/m}$$

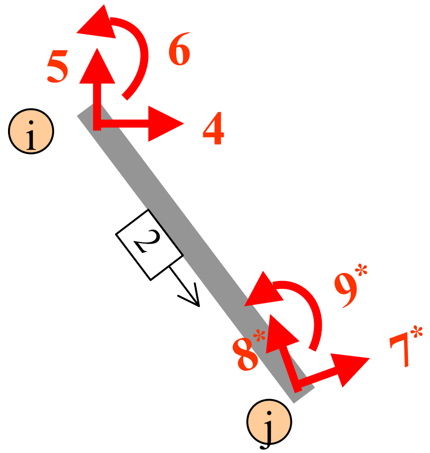


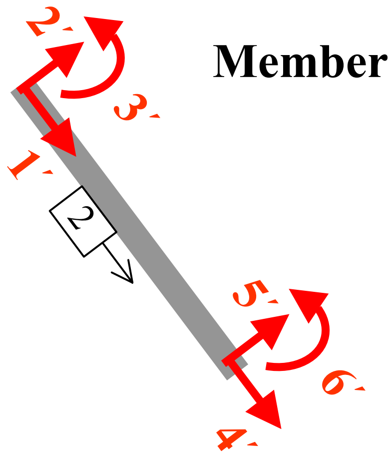




Member 1: [*q*] = [*k*][*d*] + [*q*^{F}]

$$
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = 10^3
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix} 3000 & 0 & 0 & -3000 & 0 & 0 \\ 0 & 15 & 30 & 0 & -15 & 30 \\ 0 & 30 & 80 & 0 & -30 & 40 \\ -3000 & 0 & 0 & 3000 & 0 & 0 \\ 0 & -15 & -30 & 0 & 15 & -30 \\ 0 & 30 & 40 & 0 & -30 & 80 \end{pmatrix} \end{array}
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} +
\begin{pmatrix} 0 \\ 40 \\ 26.67 \\ 0 \\ 40 \\ -26.67 \end{pmatrix}
$$





Member 2:

$$\frac{AE}{L} = \frac{(60\times10^{-3}\ \text{m}^2)(200\times10^{6}\ \text{kN/m}^2)}{3.61\ \text{m}}$$

$$= 3324\times10^{3}\ \text{kN/m}$$

$$\frac{4EI}{L} = \frac{4(200\times10^{6}\ \text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{3.61\ \text{m}}$$

$$= 88.64\times10^{3}\ \text{kN}\bullet\text{m}$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^{6}\ \text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{(3.61\ \text{m})^2}$$

$$= 36.83\times10^{3}\ \text{kN}$$

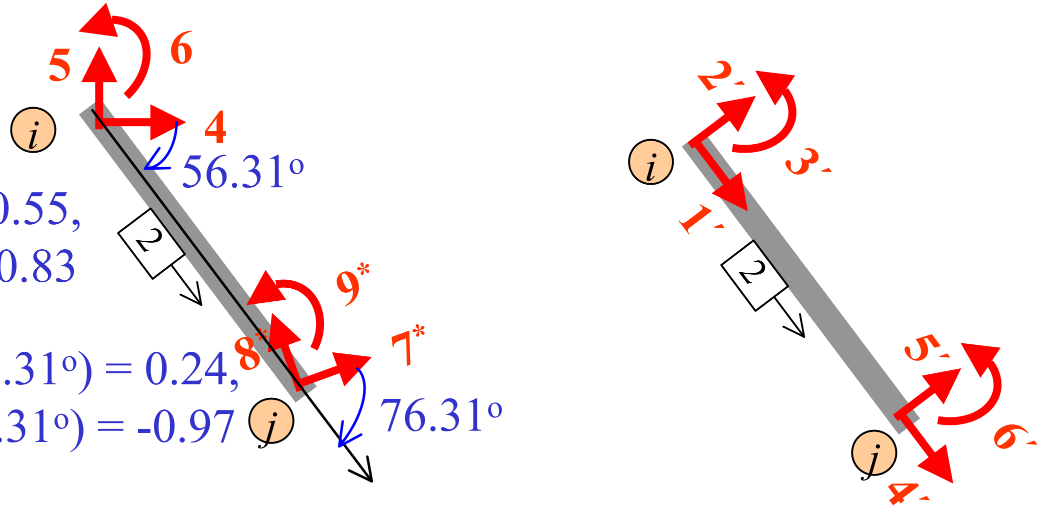
$$\frac{12EI}{L^3} = \frac{12(200\times10^{6}\ \text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{(3.61\ \text{m})^3}$$

$$= 20.41\times10^{3}\ \text{kN/m}$$

$$\frac{2EI}{L} = \frac{2(200\times10^{6}\ \text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{3.61\ \text{m}}$$

$$= 44.32\times10^{3}\ \text{kN}\bullet\text{m}$$

$$[k']_2 = 10^3 \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \\ 3324 & 0 & 0 & -3324 & 0 & 0 \\ 0 & 20.41 & 36.83 & 0 & -20.41 & 36.83 \\ 0 & 36.83 & 88.64 & 0 & -36.83 & 44.32 \\ -3324 & 0 & 0 & 3324 & 0 & 0 \\ 0 & -20.41 & -36.83 & 0 & 20.41 & -36.83 \\ 0 & 36.83 & 44.32 & 0 & -36.83 & 88.64 \end{array}$$



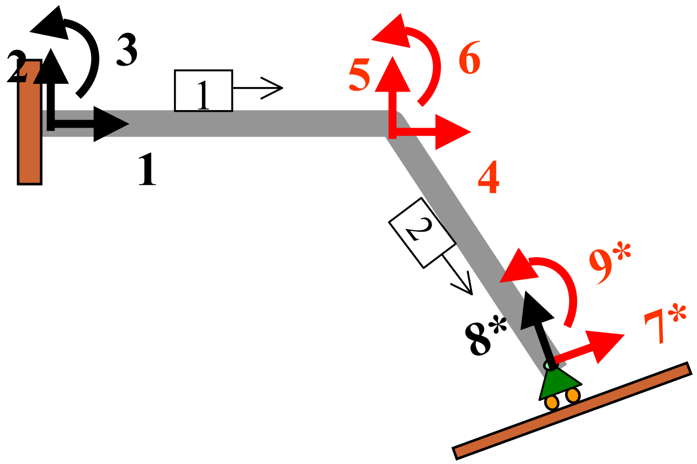
$\lambda_{ix} = \cos(-56.31^o) = 0.55,$
$\lambda_{iy} = \sin(-56.31^o) = -0.83$

$\lambda_{jx} = \cos(-76.31^o) = 0.24,$
$\lambda_{jy} = \sin(-76.31^o) = -0.97$

$[\,\boldsymbol{q}^*\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{q}'\,]$

$$
\begin{Bmatrix} q_4 \\ q_5 \\ q_6 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{Bmatrix}
=
\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7^* \\ 8^* \\ 9^* \end{array}
\begin{array}{cccccc}
1' & 2' & 3' & 4' & 5' & 6' \\
0.55 & 0.83 & 0 & 0 & 0 & 0 \\
-0.83 & 0.55 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.24 & 0.97 & 0 \\
0 & 0 & 0 & -0.97 & 0.24 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}
\begin{Bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{Bmatrix}
$$

$$
[\,k^*\,]_2 = [\,T\,]^{\mathrm{T}}[\,k'\,]_2[\,T\,] = 10^3
\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7^* \\ 8^* \\ 9^* \end{array}
\begin{array}{cccccc}
4 & 5 & 6 & 7^* & 8^* & 9^* \\
1036.923 & -1524.780 & 30.646 & -452.884 & 1787.474 & 30.646 \\
-1524.780 & 2307.582 & 20.431 & 643.585 & -2689.923 & 20.431 \\
30.646 & 20.431 & 88.643 & -35.786 & -8.717 & 44.321 \\
-452.884 & 643.585 & -35.786 & 205.452 & -759.668 & -35.786 \\
1787.474 & -2689.923 & -8.717 & -759.668 & 3139.053 & -8.717 \\
30.646 & 20.431 & 44.321 & -35.786 & -8.717 & 88.643
\end{array}
$$

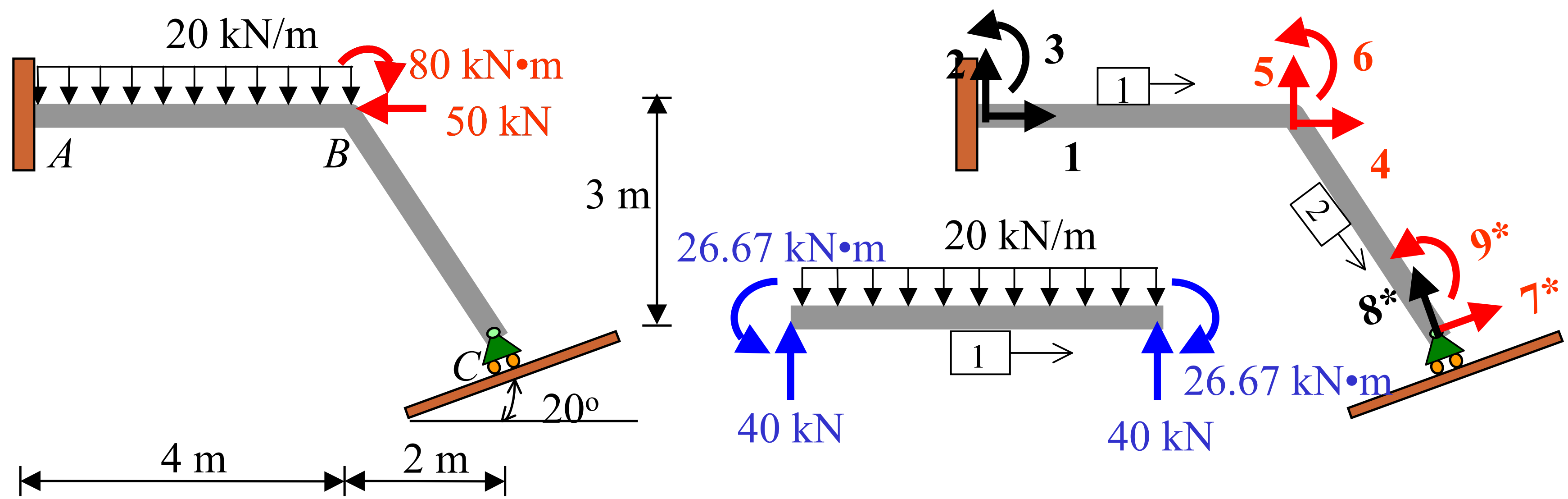


$[k]_1 = 10^3$

	1	2	3	4	5	6
1	3000	0	0	-3000	0	0
2	0	15	30	0	-15	30
3	0	30	80	0	-30	40
4	-3000	0	0	3000	0	0
5	0	-15	-30	0	15	-30
6	0	30	40	0	-30	80

$[k^*]_2 = 10^3$

	4	5	6	7*	8*	9*
4	1036.923	-1524.780	30.646	-452.884	1787.474	30.646
5	-1524.780	2307.582	20.431	643.585	-2689.923	20.431
6	30.646	20.431	88.643	-35.786	-8.717	44.321
7*	-452.884	643.585	-35.786	205.452	-759.668	-35.786
8*	1787.474	-2689.923	-8.717	-759.668	3139.053	-8.717
9*	30.646	20.431	44.321	-35.786	-8.717	88.643



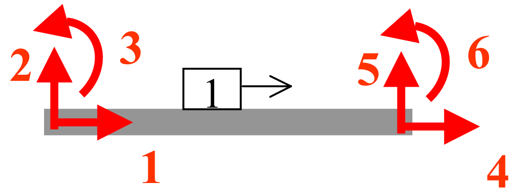
Global:

$$\begin{bmatrix} Q_4 = -50 \\ Q_5 = 0 \\ Q_6 = -80 \\ Q_{7*} = 0 \\ Q_{9*} = 0 \end{bmatrix} = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 9 \end{array} \begin{array}{ccccc} 4 & 5 & 6 & 7* & 9* \\ 4036.923 & -1524.780 & 30.646 & -452.884 & 30.646 \\ -1524.780 & 232.582 & -9.569 & 643.585 & -9.569 \\ 30.646 & -9.569 & 168.643 & -35.786 & 44.321 \\ -452.884 & 643.585 & -35.786 & 205.452 & -35.786 \\ 30.646 & 20.431 & 44.321 & -35.786 & 88.643 \end{array} \begin{bmatrix} D_4 \\ D_5 \\ D_6 \\ D_{7*} \\ D_{9*} \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \\ -26.67 \\ 0 \\ 0 \end{bmatrix}$$

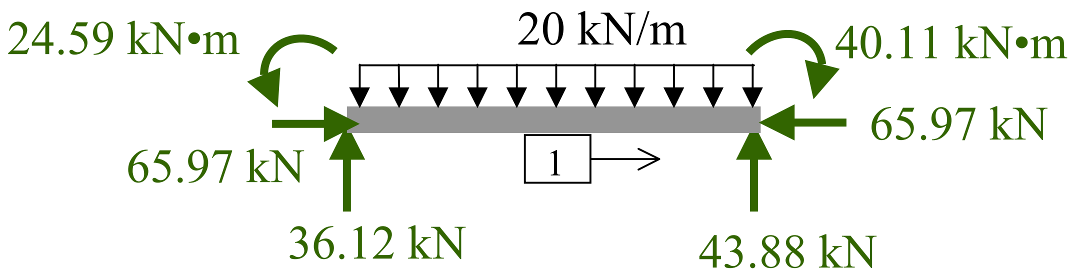
$$\begin{bmatrix} D_4 \\ D_5 \\ D_6 \\ D_{7*} \\ D_{9*} \end{bmatrix} = \begin{bmatrix} -2.199\times10^{-5} & \text{m} \\ -3.095\times10^{-4} & \text{m} \\ -2.840\times10^{-4} & \text{rad} \\ 0.979\times10^{-3} & \text{m} \\ 6.161\times10^{-4} & \text{rad} \end{bmatrix}$$

Member 1:

$$
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = 10^3
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
3000 & 0 & 0 & -3000 & 0 & 0 \\
0 & 15 & 30 & 0 & -15 & 30 \\
0 & 30 & 80 & 0 & -30 & 40 \\
-3000 & 0 & 0 & 3000 & 0 & 0 \\
0 & -15 & -30 & 0 & 15 & -30 \\
0 & 30 & 40 & 0 & -30 & 80
\end{pmatrix}
\end{array}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ D_4 \\ D_5 \\ D_6 \end{pmatrix}
+
\begin{pmatrix} 0 \\ 40 \\ 26.67 \\ 0 \\ 40 \\ -26.67 \end{pmatrix}
$$



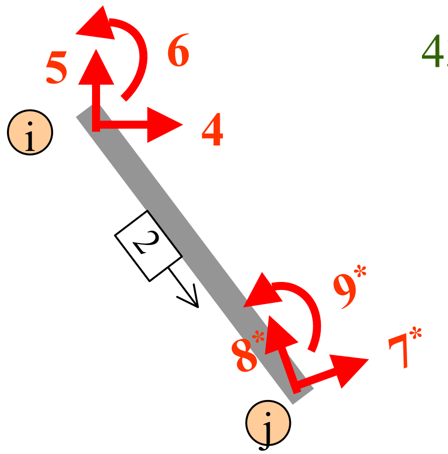
$$
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} =
\begin{pmatrix}
65.97 & \text{kN} \\
36.12 & \text{kN} \\
24.59 & \text{kN•m} \\
-65.97 & \text{kN} \\
43.88 & \text{kN} \\
-40.11 & \text{kN•m}
\end{pmatrix}
$$

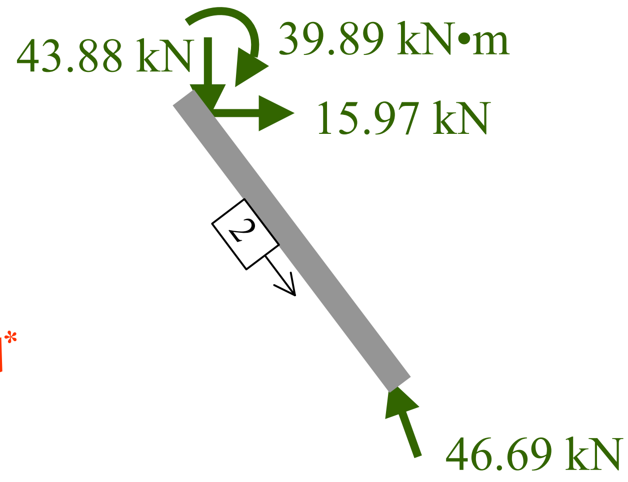


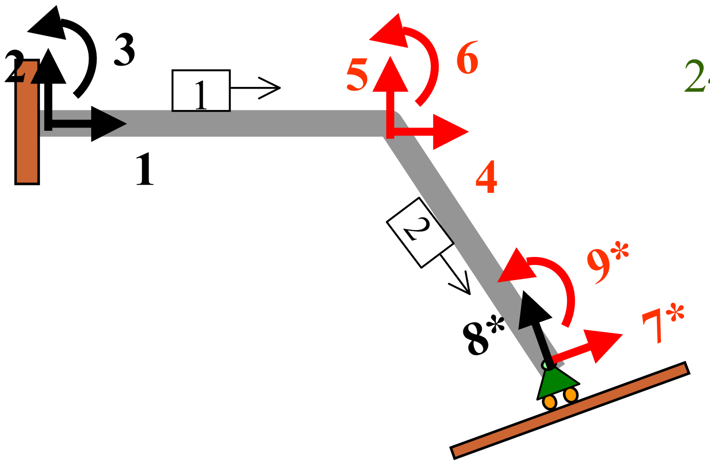
Member 2:

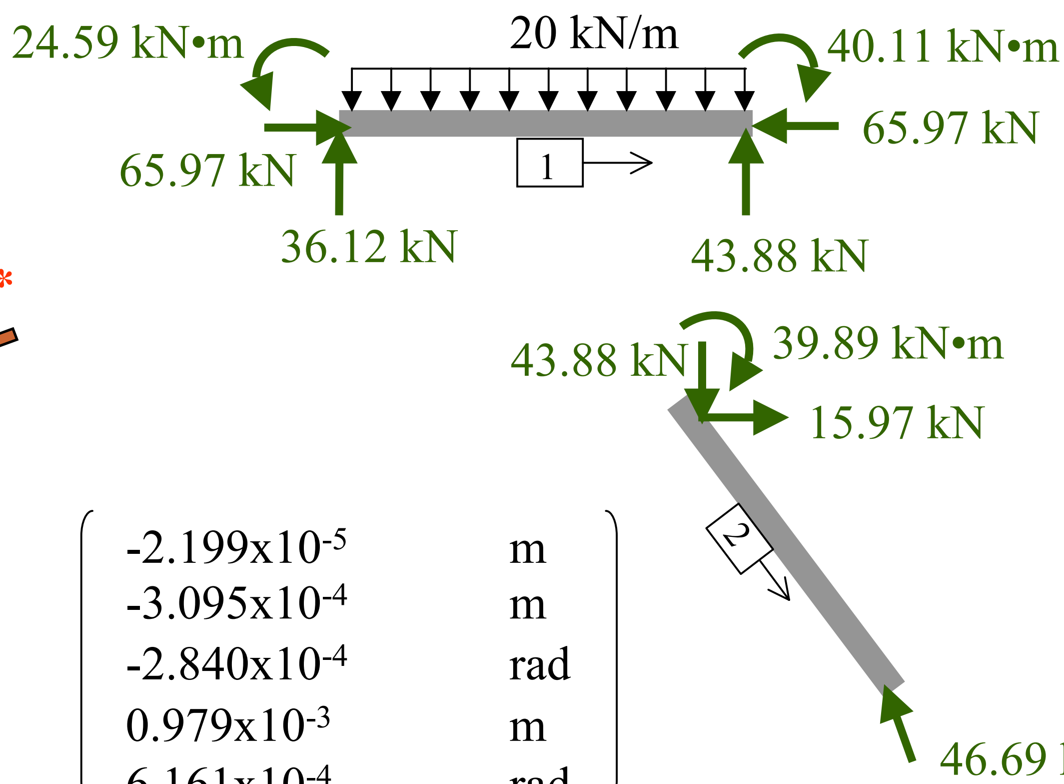
$$
\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} = 10^3
\begin{array}{c} \\ \color{red}{4} \\ \color{red}{5} \\ \color{red}{6} \\ \color{red}{7*} \\ \color{red}{8*} \\ \color{red}{9*} \end{array}
\begin{array}{c}
\begin{matrix} \color{red}{4} & \color{red}{5} & \color{red}{6} & \color{red}{7*} & \color{red}{8*} & \color{red}{9*} \end{matrix} \\
\begin{pmatrix}
1036.923 & -1508.14 & 30.57 & -455.21 & 1769.34 & 30.57 \\
-1508.14 & 2296.15 & 20.26 & 651.27 & -2678.93 & 20.26 \\
30.57 & 20.26 & 88.64 & -35.73 & -8.84 & 44.32 \\
-455.21 & 651.27 & -35.73 & 210.67 & -769.1 & -35.73 \\
1769.34 & -2678.93 & -8.84 & -769.1 & 3128.82 & -8.84 \\
30.57 & 20.26 & 44.32 & -35.73 & -8.84 & 88.64
\end{pmatrix}
\end{array}
\begin{pmatrix} D_4 \\ D_5 \\ D_6 \\ D^*_7 \\ 0 \\ D^*_9 \end{pmatrix}
$$

$$
\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} =
\begin{pmatrix}
15.97 & \text{kN} \\
-43.88 & \text{kN} \\
-39.89 & \text{kN•m} \\
0 & \text{kN} \\
46.69 & \text{kN} \\
0 & \text{kN•m}
\end{pmatrix}
$$

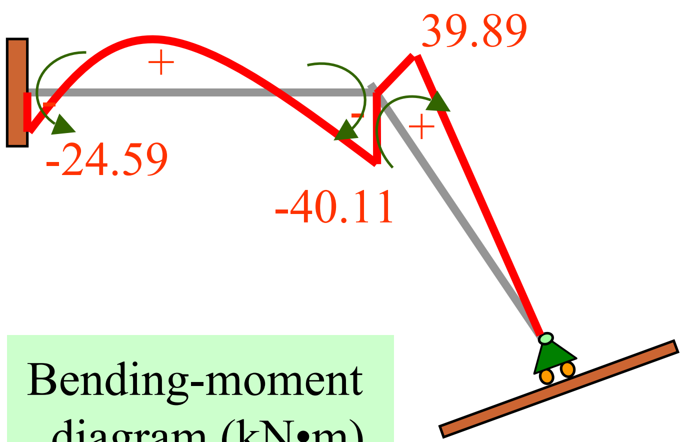




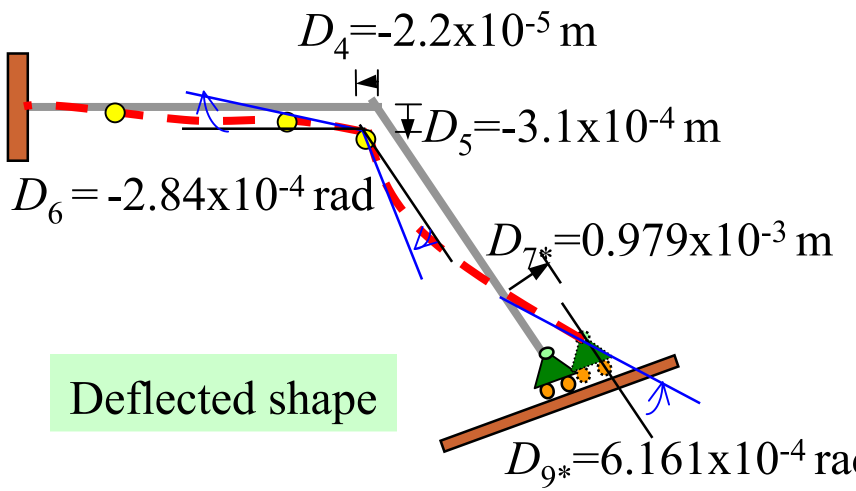




$$
\begin{pmatrix} D_4 \\ D_5 \\ D_6 \\ D_{7*} \\ D_{9*} \end{pmatrix} = \begin{pmatrix} -2.199\text{x}10^{-5} & \text{m} \\ -3.095\text{x}10^{-4} & \text{m} \\ -2.840\text{x}10^{-4} & \text{rad} \\ 0.979\text{x}10^{-3} & \text{m} \\ 6.161\text{x}10^{-4} & \text{rad} \end{pmatrix}
$$



Bending-moment diagram (kN•m)



Deflected shape